

ON THE ALGEBRAIC K -THEORY OF HIGHER CATEGORIES

I. THE UNIVERSAL PROPERTY OF WALDHAUSEN K -THEORY

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In memoriam Daniel Quillen, 1940–2011, with profound admiration.

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0. INTRODUCTION

Though in retrospect one is today able to recognize the rich prehistory of algebraic K -theory — dating back at least to Whitehead’s work on simple homotopy theory in the 1940s —, it was as a fundamental concept in Grothendieck’s celebrated reformulation of the Riemann–Roch theorem over half a century ago that algebraic K -theory made its first explicit appearance [27]. For any suitable additive subcategory \mathcal{A} of an abelian category, Grothendieck proposed a two-stage description of the abelian group that we now call $K_0(\mathcal{A})$: first, one may define the *direct sum K -theory* $K_0^\oplus(\mathcal{A})$ as the group completion of the commutative monoid of isomorphism classes of objects under direct sum; then one may define $K_0(\mathcal{A})$ as the quotient of $K_0^\oplus(\mathcal{A})$ under the relations $[E] = [E'] + [E'']$ for any short exact sequence

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

in \mathcal{A} .

Over the course of the next decade, Grothendieck and his collaborators developed a more sophisticated vision of the Riemann–Roch theorem. For this, it became necessary to describe the algebraic K -theory of suitable categories of complexes. To this end, Grothendieck and his school described the algebraic K -theory

of triangulated categories in the following manner [11]. For any essentially small triangulated category \mathcal{T} , they defined $K_0(\mathcal{T})$ as the essentially unique abelian group such that homomorphisms $K_0(\mathcal{T}) \rightarrow A$ are in bijection with such maps $\phi: \text{Obj } \mathcal{T} \rightarrow A$ that are *additive* in the sense that for any distinguished triangle

$$E' \rightarrow E \rightarrow E'' \rightarrow E'[1]$$

in \mathcal{T} , one has $\phi(E) = \phi(E') + \phi(E'')$. In other words, the map $\text{Obj } \mathcal{T} \rightarrow K_0(\mathcal{T})$ is initial among additive maps out of $\text{Obj } \mathcal{T}$.

When we pass to higher algebraic K -theory of categories, however, this pleasant state of affairs seems to dissolve. Neither Quillen's Q construction [44] nor Waldhausen's S_\bullet construction [57] yield a definition of algebraic K -theory built upon the sort of explicit universal property enjoyed by the group K_0 introduced by Grothendieck and his collaborators. Consequently, neither the K -theory of exact categories nor the K -theory of Waldhausen categories has admitted a useful *recognition principle*, and it has been something of a struggle to express exactly what information higher algebraic K -theory contains.

Our results. Our goal with this work is a first step toward repairing this situation by isolating the universal property enjoyed by the most general construction of algebraic K -theory to date. We show that Waldhausen's K -theory can be generalized to a broad class of quasicategories — which we call *Waldhausen ∞ -categories* —, and we show that this form of algebraic K -theory satisfies a simple and powerful universal property that for stable ∞ -categories reduces to the one introduced by Grothendieck and his collaborators for K_0 . Armed with this, we give new and more conceptual proofs of very general forms of the two main theorems of Waldhausen K -theory: the Additivity Theorem [57, Th. 1.4.2] (our version: Cor. 7.10.1) and the Fibration Theorem [57, Th. 1.6.4] (our version: Pr. 9.30). (In this context, Waldhausen's Approximation Theorem [57, Th. 1.6.7] is a relatively simple consequence of the fact that K -theory preserves equivalences of quasicategories; cf. Pr. 2.10.) All of these results are proved directly by means of higher categorical methods, and none of the proofs depend upon a connection to Waldhausen's setting for algebraic K -theory.

The key concept that makes this possible is the idea of a *homology theory of higher categories*: we regard Waldhausen ∞ -categories as inputs for certain kinds of homology theories, and we develop the collection of those homology theories from the ground up. Roughly speaking, we treat the homotopy theory of Waldhausen ∞ -categories as formally analogous to the category of abelian groups; to this we freely adjoin geometric realizations of simplicial objects and thus all higher homotopical information. This is the same procedure that produces the homotopy theory of (nonnegatively graded) chain complexes from abelian groups; in the case of Waldhausen ∞ -categories, we call the result *virtual Waldhausen ∞ -categories*. Now in order to study homology theories on these objects, it is necessary to determine what sort of excision axiom should be required. The form of excision we demand, *additivity*, is the splitting of exact sequences; so a homology theory with this form of excision is a functor F (valued in the ∞ -category of pointed spaces, say) with the property that for any cofiber sequence of natural transformations,

$$\Psi' \rightarrow \Psi \rightarrow \Psi''$$

one has a homotopy $F(\Psi) \simeq F(\Psi') + F(\Psi'')$, a generalization of the additivity condition used by Grothendieck et al. The main theorem of this paper, Th. 7.2, is a complete characterization of these homology theories.

We can now consider the functor $\mathcal{C} \mapsto \iota\mathcal{C}$ that assigns to any Waldhausen ∞ -category the maximal ∞ -groupoid contained therein. This is not a homology theory in the sense we have described, but we may contemplate the best approximation to this functor by a homology theory. This can be done by means of a Goodwillie derivative in the sense of [23, 25, 26], and the result is precisely Waldhausen's *connective algebraic K -theory*. Consequently, for any loop space X , the space of maps $K(\mathcal{C}) \rightarrow X$ is homotopy equivalent to the space of maps $\iota\mathcal{C} \rightarrow X$ that split exact sequences. (For a more precise statement, see Pr. 10.6.)

Overview. In more detail, let us rehearse the contents of this paper.

In the first part, we lay the groundwork for the theory of Waldhausen ∞ -categories.

- (1) We begin by introducing *pairs of ∞ -categories* (Df. 1.11), which are ∞ -categories equipped with subcategories containing all the objects; the morphisms of these subcategories will be called *cofibrations* or *ingressive morphisms*.
- (2) Waldhausen ∞ -categories (Df. 2.4) are then pairs that contain a zero object and suitable pushouts along cofibrations. In particular, an ∞ -category with a zero object and all finite colimits can be regarded

as a Waldhausen ∞ -category in which *every* morphism is ingressive; our variant of Waldhausen's Approximation Theorem (Pr. 2.10) is the observation that equivalences between these special Waldhausen ∞ -categories can be tested at the level of the homotopy category.

- (3) We then move on to study the relative theory of Waldhausen ∞ -categories — that of Waldhausen (co)cartesian fibrations, which classify functors valued in \mathbf{Wald}_∞ (Pr. 3.12). The use of these allows us to avoid solving pernicious homotopy-coherence problems; consequently, the central constructions of the paper turn out to be relatively small.
- (4) Following a spate of structural results on the collection of Waldhausen ∞ -categories (Pr. 4.3, Pr. 4.4, Pr. 4.6, and Cor. 4.7.2), we then introduce the notion of *virtual Waldhausen ∞ -categories* (Df. 4.10), which can be regarded as formal geometric realizations of simplicial diagrams of Waldhausen ∞ -categories. We give an explicit construction of this formal geometric realization (Cnstr. 4.15).

In the second part, we study additive theories for Waldhausen categories.

- (5) We begin with a discussion of *filtered objects* in Waldhausen ∞ -categories (Df. 5.3), which are sequences of cofibrations of finite length. We equip the collection of filtered objects in a Waldhausen ∞ -category with a somewhat subtle pair structure (Df. 5.6); with this pair structure, the forgetful functor to $N\Delta^{\text{op}}$ given by length is a Waldhausen cocartesian fibration (Pr. 5.7). Among the filtered objects are the *totally filtered objects* (Df. 5.10), which are those sequences of cofibrations of finite length that begin at zero. The collection of totally filtered objects of a Waldhausen ∞ -category inherits a pair structure from the ∞ -category of all filtered objects, and with this pair structure, we show, by a rather delicate argument, that the forgetful functor to $N\Delta^{\text{op}}$ given by length is a Waldhausen cocartesian fibration (Th. 5.16). After passing to formal geometric realizations, we find that the virtual Waldhausen ∞ -category of filtered objects plays the role of a *cone* on a Waldhausen ∞ -category (Pr. 5.21).
- (6) Then, in order to regard the virtual Waldhausen ∞ -category of totally filtered objects as a *suspension* of a Waldhausen ∞ -category, it becomes necessary to pass to a localization of the ∞ -category of virtual Waldhausen ∞ -categories. The objects of this localization are called *distributive* (Df. 6.2). It turns out that the virtual Waldhausen ∞ -category of totally filtered objects of a Waldhausen ∞ -category is automatically distributive (Pr. 6.10), and consequently formation of totally filtered objects is indeed a suspension functor in this context (Cor. 6.10.1).
- (7) Now we may speak of *theories* of Waldhausen ∞ -categories, which are pointed, continuous functors from Waldhausen ∞ -categories to pointed spaces (or pointed objects of another ∞ -topos). We prove our main theorem (Th. 7.2), which gives a collection of equivalent characterizations of *additive theories*, the most important of which is the condition that the theory factor through an excisive functor on distributive virtual Waldhausen ∞ -categories. As a result of this, it turns out that the ∞ -category of additive theories can be identified with the ∞ -category of such excisive functors (Th. 7.4), and, thanks to the machinery of the Goodwillie calculus, any theory admits a best additive approximation (Th. 7.6), called its *additivization*. Since totally filtered objects act as suspension, this additivization can be constructed as $\Phi \mapsto \text{colim}_n \Omega^n \circ \Phi \circ \mathcal{S}^n$, but under mild hypotheses on Φ , the colimit is not necessary (Cor. 7.10.1); in these cases, the additivization of Φ can be computed as $\Omega \circ \Phi \circ \mathcal{S}$. This is our analogue of Waldhausen's Additivity Theorem, and it gives a local universal property satisfied by the additivization (Pr. 7.11).
- (8) Now we can collect the low-hanging fruit from additivity. We use the Eilenberg swindle to show that infinite coproducts in a Waldhausen ∞ -category make it invisible to additive theories (Pr. 8.1), and we show that for Waldhausen ∞ -categories with the *maximal* pair structure, stabilization does not affect its value under additive functors (Pr. 8.2). By analyzing the additivization of the Yoneda embedding, we find that the ∞ -category of distributive virtual Waldhausen ∞ -categories has the unusual property that $\Omega\Sigma$ is the Goodwillie differential of the identity. We then analyze, for any exact functor ψ of Waldhausen ∞ -categories, a construction whose value under any additive functor is the cofiber of the map induced by ψ ; this is the first Fibration Theorem (Th. 8.9). Armed with this, we give a necessary and sufficient condition on such an exact functor that ψ is carried to an equivalence under any additive theory.
- (9) In analogy with Waldhausen's theory of categories with cofibrations and weak equivalences, we introduce labeled Waldhausen ∞ -categories (Df. 9.2), and labeled Waldhausen (co)cartesian fibrations (Df. 9.6), and we show that filtered and totally filtered objects in a labeled Waldhausen ∞ -category inherit a labeling over $N\Delta^{\text{op}}$ (Pr. 9.10 and Pr. 9.11). To any labeled Waldhausen ∞ -category, we can attach a

virtual Waldhausen ∞ -category (Cnstr. 9.12), which permits us to apply theories to labeled Waldhausen ∞ -categories. We briefly discuss the possibility of formally inverting the labeled morphisms (in the ∞ -categorical sense), and we give two examples in which we can recognize the resulting Waldhausen ∞ -category (Pr. 9.21 and Pr. 9.22). We then prove a variant of Waldhausen’s Generic Fibration Theorem (Th. 9.30) that, for any labeled Waldhausen ∞ -category $(\mathcal{A}, w\mathcal{A})$ and any additive theory Φ , exhibits a cofiber sequence relating the theory applied to the virtual Waldhausen ∞ -category attached to \mathcal{A} , to \mathcal{A} itself, and to the full subcategory of \mathcal{A} spanned by those objects that are w -equivalent to zero.

In the final part, we apply our machinery to the study of algebraic K -theory and further to four relatively easy examples of Waldhausen ∞ -categories.

- (10) Algebraic K -theory is defined (Df. 10.1) as the additivization of the functor ι that carries any Waldhausen ∞ -category to the maximal Kan complex contained therein. Since ι is representable by the nerve of Segal’s category Γ^{op} (Pr. 10.4), it follows that the space of natural transformations from K -theory to an additive theory Φ is naturally equivalent to $\Phi(N\Gamma^{\text{op}})$ (Cor. 10.4.1). We can also use the universal property of the additivization to deduce comparison theorems (Cor. 10.5.2, Cor. 10.8.1, and Cor. 10.8.2).
- (11) As a first application, we compute the algebraic K -theory of the total space \mathcal{X}^{\otimes} of a symmetric monoidal ∞ -groupoid \mathcal{X} . The result turns out to be equivalent to a group completion of a *wreath product* of \mathcal{X} with the nerve of the ordinary category of pointed finite sets (Pr. 11.17).
- (12) Next, we lay the groundwork necessary to extend Waldhausen’s A -theory of spaces to more general ∞ -topoi, but we only scratch the surface of what is possible. Our construction is covariantly functorial in arbitrary maps of the ∞ -topos, and it is contravariantly functorial in maps that satisfy a certain finiteness condition (Df. 12.10); these satisfy a base change compatibility (Pr. 12.9). This A -theory coincides with the K -theory of a corresponding ∞ -category of spectra (Pr. 12.13), and we remark that, in effect, our construction of A -theory already contains all the assembly morphisms (12.15). Finally, we note that, when regarded as a functor of all suitable ∞ -topoi at once, A -theory admits a rich structure (Pr. 12.22).
- (13) We can also apply our foundational work to the study of the connective K -theory of E_1 -algebras in suitable ground ∞ -categories. We define a notion of a *perfect* left module over an E_1 -algebra (Df. 13.2). If the ground ∞ -category satisfies a certain technical condition, which we call *admissibility* (Df. 13.8), then the ∞ -category of perfect left modules over any E_1 -algebra can be endowed with a pair structure that makes it into a Waldhausen ∞ -category. We then define the K -theory of the E_1 -algebra as the K -theory of the of this Waldhausen ∞ -category. In the special case of an E_1 ring spectrum Λ , for any set S of homogenous elements of $\pi_*\Lambda$ that satisfies a left Ore condition, we obtain a cofiber sequence

$$K(\mathbf{Nil}_{(\Lambda, S)}^{\omega}) \longrightarrow K(\Lambda) \longrightarrow K(\Lambda[S^{-1}]),$$

in which the first term is the K -theory of the ∞ -category of S -nilpotent perfect Λ -modules (Pr. 13.16). Such a result is surely well-known among experts; see for example [13, Pr. 1.4 and Pr. 1.5].

- (14) Finally, we apply our construction to ∞ -categories of perfect modules over quasicompact nonconnective spectral Deligne–Mumford stacks. We are forced to extend some of the basic definitions and structural results on quasicohherent modules to the setting of nonconnective spectral Deligne–Mumford stacks, but once we do, we are able to define the algebraic K -theory of these stacks (Df. 14.10). After applying our Special Fibration Theorem in this context, the result is an analog of what Thomason called the “proto-localization” theorem [51, Th. 5.1], valid in this context of spectral algebraic geometry (Pr. 14.13); this is a cofiber sequence of connective spectra

$$\mathbf{K}^{\text{conn}}(\mathcal{X} \setminus \mathcal{U}) \longrightarrow \mathbf{K}^{\text{conn}}(\mathcal{X}) \longrightarrow \mathbf{K}^{\text{conn}}(\mathcal{U})$$

corresponding to an open immersion $j: \mathcal{U} \longrightarrow \mathcal{X}$. Here $\mathbf{K}^{\text{conn}}(\mathcal{X} \setminus \mathcal{U})$ is the K -theory of the ∞ -category perfect modules \mathcal{M} on \mathcal{X} such that $j^*\mathcal{M} \simeq 0$. Our proof is new even in the setting originally contemplated Thomason (at least for quasicompact schemes), and it also provides a new proof of the localization theorem of Krishna–Østvær [35, Th. 3.7].

Relation to other work. Our universal characterization of algebraic K -theory has probably been known — perhaps in a more restrictive setting and certainly in a different language — to a variety of experts for many years. In fact, the universal property stated here has endured a lengthy gestation: the first version of this characterization emerged during a question-and-answer session between the author and John Rognes after a talk given by the author at the University of Oslo in 2006.

But long before that, of course, came the foundational work of Waldhausen [57]. Since it is known today that relative categories comprise a model for the homotopy theory of ∞ -categories [4], the work of Waldhausen can be said to represent the first study of the algebraic K -theory of higher categories. Furthermore, the idea that the defining property of this algebraic K -theory is additivity is strongly suggested by Waldhausen, and this point is driven home in the work of Randy McCarthy [43] and Ross Staffeldt [49], both of whom recognized long ago that the additivity theorem is the ur-theorem of algebraic K -theory.

The idea that algebraic K -theory might itself be expressible as a suitable Goodwillie derivative was directly inspired by the ICM talk of Tom Goodwillie [24] and the remarkable flurry of research into the relationship between algebraic K -theory and the calculus of functors — though of course the setting for our Goodwillie derivative is more primitive than the one studied by Goodwillie et al.

More recently, Bertrand Toën and Gabriele Vezzosi showed [54] that the Waldhausen K -theory of many of the best-known examples of Waldhausen categories is in fact an invariant of the simplicial localization; thus Toën and Vezzosi are more explicit in identifying higher categories as a natural domain for K -theory. In fact, in the final section of [54], the authors suggest a strategy for constructing directly the K -theory of a Segal category by means of an “ S_\bullet construction up to coherent homotopy.” The desired properties of their construction are reflected precisely in our construction \mathcal{S} . These insights were explored more deeply in the work of Blumberg and Mandell [14]; they give an explicit description of Waldhausen’s S_\bullet construction in terms of the mapping spaces of the simplicial localization, and they extend Waldhausen’s approximation theorem to show that in many cases, equivalences of homotopy categories alone are enough to deduce equivalences of K -theory spectra.

Even more recent work of Andrew Blumberg, David Gepner, and Gonalo Tabuada [12] has built upon brilliant work of the last of these authors in the context of DG categories [50] to produce another universal characterization of the algebraic K -theory of stable ∞ -categories. One of their main results may also be summarized by saying that the algebraic K -theory of ∞ -categories is the universal additive invariant; however, such a summary belies the substantial differences between the present work and theirs.

- (1) First, the *setting* of our results is different. In [12], the authors concentrate on stable ∞ -categories. Consequently, their results do not apply *out of the box* to characterize algebraic K -theory in more classical contexts. For example, if one wishes to regard Quillen’s algebraic K -theory of exact categories as an instance of their universal K -theory, one must apply a result such as [51, Th 1.11.7] in order to enlarge the scope of their result to handle these cases. (The two will in general differ in degree 0, because no distinction is made in [12] between the K -theory of a stable ∞ -category and the K -theory of its idempotent completion.) Their universality result does not seem to apply readily to more exotic Waldhausen categories that do not admit cylinder functors, such as those that appear in Zakharevich’s work on higher scissors congruence [59] or the Waldhausen categories of spaces in which the weak equivalences are simple maps, as in [57, Part 3] or [58]. On the other hand, Blumberg, Gepner, and Tabuada also study nonconnective deloopings of K -theory, with which we do not contend here.
- (2) Moreover, the *nature* of our universal property is different. Blumberg, Gepner, and Tabuada embed the collection of stable ∞ -categories into a symmetric monoidal, stable ∞ -category \mathcal{M}_{add} of *noncommutative motives* in such a way as to guarantee that additive functors of stable ∞ -categories valued in spectra can be identified with colimit-preserving functors on \mathcal{M}_{add} . From this perspective, algebraic K -theory is corepresentable by the “unit motive.” By contrast, our ∞ -category $\mathbf{V}_{\text{add}} \mathbf{Wald}_\infty$ is somewhat smaller: objects therein can be regarded as certain *formal geometric realizations* of simplicial Waldhausen ∞ -categories, and additive functors valued in spaces may be identified with excisive functors on $\mathbf{V}_{\text{add}} \mathbf{Wald}_\infty$. From this perspective, algebraic K -theory may be identified with the linearization (in the sense of Goodwillie) of the functor that assigns to any Waldhausen ∞ -category the maximal ∞ -groupoid contained therein. An immediate consequence is that additive functors admit unique, functorial, connective deloopings. Our result permits us to recover a slightly different (but more general) representability theorem for algebraic K -theory.
- (3) Finally, the *structure of our argument* is different. The work of [12] ultimately relies upon delicate (and beautiful) strictification arguments. Blumberg, Gepner, and Tabuada define connective algebraic K -theory by means of an ∞ -categorical variant of Waldhausen’s S_\bullet construction, but in order to study this K -theory (and in particular to verify their universal property), they must compare their construction to the classical S_\bullet -construction applied to strict, spectrally-enriched Waldhausen categories cooked up from

stable ∞ -categories using Morita theory. By contrast, our construction remains firmly in the realm of ∞ -categories. We develop a theory of additive functors on Waldhausen ∞ -categories; we *define* algebraic K -theory as an additive functor with a universal property; and we prove the basic theorems of algebraic K -theory (the Additivity, Fibration, and Approximation Theorems) using purely ∞ -categorical arguments. It is only *after* this work that we prove that our algebraic K -theory functor extends the classical one (Cor. 10.8.3).

Finally, we recall that Waldhausen’s formalism for algebraic K -theory has of course been applied in the context of associative S -algebras by Elmendorf, Kriz, Mandell, and May [22], and in the context of schemes and algebraic stacks by Thomason and Trobaugh [51], Toën [52], Joshua [29, 30, 31], and others. The applications of the last two sections of this paper are of course nothing more than extensions of their work.

A word on higher categories. We use the theory of $(\infty, 1)$ -categories — or, more briefly, ∞ -categories — in this paper. In particular, we use the *quasicategory* model of ∞ -categories. Quasicategories were invented in the 1970s by Boardman and Vogt [15], who called them *weak Kan complexes*, and they were studied extensively by Joyal [32, 33] and Lurie [37]. We emphasize that quasicategories are but one of an array of equivalent models of ∞ -categories (including simplicial categories [20, 18, 19, 7], Segal categories [28, 6, 48], complete Segal spaces [45, 9]), and there is no doubt that the results here could be satisfactorily proved in any one of these models. Indeed, there is a canonical equivalence between any two of these homotopy theories [34, 8, 10] (or any other homotopy theory that satisfies the axioms of [53] or of [5]), through which one can surely translate the main theorems here into theorems in the language of any other model.

That said, we wish to emphasize that we employ many of the technical details of the particular theory of quasicategories as presented in [37] in a critical way in this paper. In particular, the theory of fibrations, developed by Joyal and presented in Chapter 2 of [37], is vital to our work here, as it provides a convenient language in which to formalize the notion of “pseudofunctors” valued in the ∞ -category of ∞ -categories or of ∞ -groupoids. Indeed, it is the convenience and relative simplicity of this theory that compelled us to work with this model.

Acknowledgments. There are a lot of people to thank. Without the foundational work of André Joyal and Jacob Lurie on quasicategories, the results here would not admit such simple statements or such straightforward proofs. I thank Jacob Lurie also for patiently and generously answering a number of (often naïve) questions during the course of the work represented here. My conversations with Andrew Blumberg over the past few years have been consistently enlightening, and I suspect that a number of the results here amount to elaborations of insights he had long ago. I have also benefitted from conversations with Dan Kan.

In the spring of 2012, I gave a course at MIT on the subject of this paper. During that time, several sharp-eyed students spotted errors, including especially Rune Haugseng, Luis Alexandre Pereira, and Guozhen Wang. I owe them my thanks for their scrupulousness.

John Rognes has declined to be listed as a coauthor, but his influence on this work has been tremendous. He was present at the conception of the main result, and this paper is teeming with insights I inherited from him.

On a more personal note, I thank Alexandra Sear for her unfailing patience and support during this paper’s ridiculously protracted writing process, and I thank Sumner McKane for perfect soundtracks for thinking about K -theory.

Part 1. Pairs and Waldhausen ∞ -categories

In this part, we introduce the basic inputs for additive and localizing functors, including the various forms of K -theory we study. We begin with the notion of a *pair* of ∞ -categories, which is nothing more than an ∞ -category with a subcategory of “cofibrations” that contains the equivalences. Among the pairs of ∞ -categories, we will then isolate the *Waldhausen ∞ -categories* as the input for algebraic K -theory; these are pairs that contain a zero object and pushouts along all cofibrations. We also introduce a relative notion of Waldhausen ∞ -category, namely *Waldhausen (co)cartesian fibrations*, which classify functors valued in the ∞ -category \mathbf{Wald}_∞ of Waldhausen ∞ -categories. Next, we study limits and colimits in \mathbf{Wald}_∞ , and we

construct the ∞ -category of *virtual Waldhausen ∞ -categories*, whose homotopy theory serves as the basis for all the work we do in this paper.

1. PAIRS OF ∞ -CATEGORIES

Here we introduce the basic notion of a *pair* of ∞ -categories, which is nothing more than an ∞ -category with a subcategory that contains the equivalences. Among the pairs of ∞ -categories, we will isolate the Waldhausen ∞ -categories as the input for algebraic K -theory.

First, however, we must recall some elements of the theory of ∞ -categories. In particular, in order to circumvent the set-theoretic difficulties arising from the consideration of categories of categories and the like, we must employ some artifice. Hence to the usual Zermelo–Frankel axioms ZFC of set theory (including the Axiom of Choice) we add the following *Universe Axiom* of Grothendieck and Verdier. The resulting set theory, called ZFCU, will be employed in this paper.

1.1. **Axiom** (Universe). Any set is an element of a universe.

1.2. This axiom is independent of the others of ZFC, since any universe \mathbf{U} is itself a model of Zermelo–Frankel set theory. Equivalently, we assume that for any cardinal τ , there exists a strongly inaccessible cardinal κ with $\tau < \kappa$; for any strongly inaccessible cardinal κ , the set \mathbf{V}_κ of sets whose rank is strictly smaller than κ is a universe.

1.3. **Notation.** In addition, we fix, once and for all, three uncountable strongly inaccessible cardinals $\kappa_0 < \kappa_1 < \kappa_2$ and the corresponding universes $\mathbf{V}_{\kappa_0} \in \mathbf{V}_{\kappa_1} \in \mathbf{V}_{\kappa_2}$. Now a set, simplicial set, category, etc., will be said to be *small* if it is contained in the universe \mathbf{V}_{κ_0} ; it will be said to be *large* if it is contained in the universe \mathbf{V}_{κ_1} ; and it will be said to be *huge* if it is contained in the universe \mathbf{V}_{κ_2} .

We will use the language and results of [37] systematically, and we will occasionally refer also to [41]. Here, let us fix some useful notation.

1.4. **Notation.** Simplicial categories (i.e., categories enriched in the category of spaces) will frequently be denoted with a superscript $(-)^{\Delta}$. To refer to the underlying ordinary category (given by taking the 0-simplices of the Mor spaces), we will change the superscript to $(-)^0$. If the Mor spaces are fibrant, then to refer to the corresponding ∞ -category (given by taking the simplicial nerve N [37, Df. 1.1.5.5]), we will drop the superscript altogether.

It will also be convenient to have a model of various ∞ -categories as *relative categories* [4]. To make this precise, we recall the following.

1.5. **Definition.** A *relative category* is an ordinary category C along with a subcategory wC that contains all the identity maps of C . The maps of wC will be called *weak equivalences*. A *relative functor* $(C, wC) \rightarrow (D, wD)$ is a functor that carries wC to wD .

Suppose (C, wC) a relative category. An ∞ -category A equipped with a functor $NC \rightarrow A$ will be said to be a *relative nerve* of (C, wC) if it satisfies the following universal property. For any ∞ -category B , the induced functor

$$\mathrm{Fun}(A, B) \rightarrow \mathrm{Fun}(NC, B)$$

is fully faithful, and its essential image is the full subcategory spanned by those functors $NC \rightarrow B$ that carry the edges of wC to equivalences in B .

1.6. **Notation.** For any ∞ -category A , there exists a simplicial subset $\iota A \subset A$, which is the largest Kan simplicial subset of A [37, 1.2.5.3]; this is the smallest simplicial subset containing the equivalences of A . We shall call this space the *interior ∞ -groupoid* of A . The assignment $A \mapsto \iota A$ defines a right adjoint ι to the inclusion functor u from Kan simplicial sets to ∞ -categories.

1.7. There are several functorial constructions of a relative nerve of a relative category (C, wC) , all of which are (necessarily) equivalent.

(1.7.1) One may form the hammock localization $L^H(C, wC)$ [18]; then a relative nerve can be constructed as the simplicial nerve of the natural functor $C \rightarrow R(L^H(C, wC))$, where R denotes any fibrant replacement for the Bergner model structure [7].

- (1.7.2) One may mark the edges of NC that correspond to weak equivalences in C to obtain a marked simplicial set [37, §3.1]; then one may use the cartesian model structure on marked simplicial sets (over Δ^0) to find a marked anodyne morphism

$$(NC, NwC) \longrightarrow (N(C, wC), \iota N(C, wC)),$$

wherein $N(C, wC)$ is an ∞ -category. With this map, the ∞ -category $N(C, wC)$ is then a relative nerve.

- (1.7.3) A relative nerve can be constructed as a fibrant model of the homotopy pushout in the Joyal model structure on simplicial sets of the map

$$\coprod_{\phi \in wC} \Delta^1 \longrightarrow \coprod_{\phi \in wC} \Delta^0$$

along the map $\coprod_{\phi \in wC} \Delta^1 \longrightarrow NC$.

1.8. Notation. The large simplicial category \mathbf{Kan}^Δ is the category of small Kan simplicial sets, with the usual notion of mapping space. The large simplicial category $\mathbf{Cat}_\infty^\Delta$ is defined in the following manner [37, Df. 3.0.0.1]. The objects of $\mathbf{Cat}_\infty^\Delta$ are small ∞ -categories, and for any two ∞ -categories A and B , the morphism space

$$\mathbf{Cat}_\infty^\Delta(A, B) := \iota \text{Fun}(A, B)$$

is the largest Kan simplicial subset contained in the ∞ -category $\text{Fun}(A, B)$.

Similarly, we may define the huge simplicial category $\mathbf{Kan}(\kappa_1)^\Delta$ of large simplicial sets and the huge simplicial category $\mathbf{Cat}_\infty(\kappa_1)^\Delta$ of large ∞ -categories.

1.9. Denote by $w\mathbf{Kan}^0 \subset \mathbf{Kan}^0$ the subcategory consisting of weak equivalences of simplicial sets. Then \mathbf{Kan} is a relative nerve of $(\mathbf{Kan}^0, w\mathbf{Kan}^0)$. Similarly, if one denotes by $w\mathbf{Cat}_\infty^0 \subset \mathbf{Cat}_\infty^0$ along with the subcategory of categorical equivalences of ∞ -categories, then \mathbf{Cat}_∞ is a relative nerve of $(\mathbf{Cat}_\infty^0, w\mathbf{Cat}_\infty^0)$. This follows directly from [37, Pr. 3.1.3.5, Pr. 3.1.3.7, Cor. 3.1.4.4].

Since the functors u and ι preserve weak equivalences, they give rise to an adjunction of ∞ -categories

$$u: \mathbf{Kan} \rightleftarrows \mathbf{Cat}_\infty: \iota.$$

1.10. Recall [37, Rm. 1.2.11] that a *subcategory* of an ∞ -category A is a simplicial subset $A' \subset A$ such that for some subcategory $(hA)'$ of the homotopy category hA ,

$$\begin{array}{ccc} A' & \hookrightarrow & A \\ \downarrow & & \downarrow \\ N(hA)' & \hookrightarrow & N(hA) \end{array}$$

is a pullback diagram of simplicial sets. In particular, note that a subcategory of an ∞ -category is specified uniquely by specifying a subcategory of its homotopy category. Note also that any inclusion $A' \hookrightarrow A$ of a subcategory is an inner fibration.

We will say that A' is *stable under equivalences* if the subcategory $(hA)' \subset hA$ above can be chosen to be stable under isomorphisms.

Now we are prepared to describe the notion of a *pair* of ∞ -categories.

1.11. Definition. (1.11.1) By a *pair* $(\mathcal{C}, \mathcal{C}_\dagger)$ of ∞ -categories (or simply a *pair*), we shall mean an ∞ -category \mathcal{C} coupled with a subcategory $\mathcal{C}_\dagger \subset \mathcal{C}$ containing the maximal Kan complex $\iota\mathcal{C} \subset \mathcal{C}$. Morphisms of \mathcal{C}_\dagger will be said to be *ingressive* morphisms or *cofibrations*.

(1.11.2) A *functor of pairs* $\psi: \mathcal{C} \longrightarrow \mathcal{D}$ is a commutative diagram

$$(1.11.3) \quad \begin{array}{ccc} \mathcal{C}_\dagger & \xrightarrow{\psi_\dagger} & \mathcal{D}_\dagger \\ \downarrow & & \downarrow \\ \mathcal{C} & \xrightarrow{\psi} & \mathcal{D} \end{array}$$

of ∞ -categories.

(1.11.4) A functor of pairs $\mathcal{C} \rightarrow \mathcal{D}$ is said to be *strict* if the diagram (1.11.3) is a pullback diagram in \mathbf{Cat}_∞ .

(1.11.5) A *subpair* of a pair $(\mathcal{C}, \mathcal{C}_\dagger)$ is a subcategory $\mathcal{D} \subset \mathcal{C}$ equipped with a pair structure $(\mathcal{D}, \mathcal{D}_\dagger)$ such that the inclusion $\mathcal{D} \hookrightarrow \mathcal{C}$ is a strict functor of pairs.

1.12. **Notation.** (1.12.1) For any two pairs \mathcal{C} and \mathcal{D} , we denote by $\mathrm{Fun}_{\mathbf{Pair}_\infty}^\#(\mathcal{C}, \mathcal{D})$ the ∞ -category of functors of pairs defined by the formula

$$\mathrm{Fun}_{\mathbf{Pair}_\infty}^\#(\mathcal{C}, \mathcal{D}) := \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \times_{\mathrm{Fun}(\mathcal{C}_\dagger, \mathcal{D}_\dagger)} \mathrm{Fun}(\mathcal{C}_\dagger, \mathcal{D}_\dagger).$$

(1.12.2) For any two pairs \mathcal{C} and \mathcal{D} , we denote by $\mathrm{Fun}_{\mathbf{Pair}_\infty}^b(\mathcal{C}, \mathcal{D})$ the full subcategory of the ∞ -category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by the functors $\mathcal{C} \rightarrow \mathcal{D}$ that carry cofibrations to cofibrations.

(1.12.3) The large simplicial category $\mathbf{Pair}_\infty^\Delta$ is defined in the following manner. The objects of $\mathbf{Pair}_\infty^\Delta$ are small pairs of ∞ -categories, and for any two pairs of ∞ -categories \mathcal{C} and \mathcal{D} , the morphism space $\mathbf{Pair}_\infty^\Delta(\mathcal{C}, \mathcal{D})$ is given by the formula

$$\begin{aligned} \mathbf{Pair}_\infty^\Delta(\mathcal{C}, \mathcal{D}) &:= \iota \mathrm{Fun}_{\mathbf{Pair}_\infty}^\#(\mathcal{C}, \mathcal{D}) \\ &\cong \mathbf{Cat}_\infty^\Delta(\mathcal{C}, \mathcal{D}) \times_{\mathbf{Cat}_\infty^\Delta(\mathcal{C}_\dagger, \mathcal{D}_\dagger)} \mathbf{Cat}_\infty^\Delta(\mathcal{C}_\dagger, \mathcal{D}_\dagger). \end{aligned}$$

(1.12.4) Suppose $(\mathcal{C}, \mathcal{C}_\dagger)$ a pair. Then a cofibration will frequently be denoted by an arrow with a tail: \rightarrowtail . We will often abuse notation by simply writing \mathcal{C} for the pair $(\mathcal{C}, \mathcal{C}_\dagger)$.

Now the ∞ -category \mathbf{Pair}_∞ is the simplicial nerve of this simplicial category (Nt. 1.4). There is a natural forgetful functor $u: \mathbf{Pair}_\infty \rightarrow \mathbf{Cat}_\infty$, which identifies the mapping spaces of \mathbf{Pair}_∞ with certain connected components of the mapping spaces of \mathbf{Cat}_∞ .

1.13. **Lemma.** *For any small pairs \mathcal{C} and \mathcal{D} , the forgetful functor $u: \mathbf{Pair}_\infty \rightarrow \mathbf{Cat}_\infty$ induces a (homotopy) monomorphism*

$$\mathbf{Pair}_\infty^\Delta(\mathcal{C}, \mathcal{D}) \rightarrow \mathbf{Cat}_\infty^\Delta(\mathcal{C}, \mathcal{D})$$

in the ∞ -category \mathbf{Kan} .

Proof. It is enough to check that the map

$$\mathbf{Cat}_\infty^\Delta(\mathcal{C}_\dagger, \mathcal{D}_\dagger) \rightarrow \mathbf{Cat}_\infty^\Delta(\mathcal{C}_\dagger, \mathcal{D})$$

is a monomorphism of \mathbf{Kan} . For this, suppose $m > 0$, and suppose we have a square

$$\begin{array}{ccc} \partial\Delta^m & \longrightarrow & \mathbf{Cat}_\infty^\Delta(\mathcal{C}_\dagger, \mathcal{D}_\dagger) \\ \downarrow & & \downarrow \\ \Delta^m & \longrightarrow & \mathbf{Cat}_\infty^\Delta(\mathcal{C}_\dagger, \mathcal{D}). \end{array}$$

This is tantamount to a diagram

$$\begin{array}{ccc} \partial\Delta^m \times \mathcal{C}_\dagger & \longrightarrow & \mathcal{D}_\dagger \\ \downarrow & & \downarrow \\ \Delta^m \times \mathcal{C}_\dagger & \longrightarrow & \mathcal{D}, \end{array}$$

in which, for any vertex $x \in \mathcal{C}_0$, the maps $\partial\Delta^m \times \{x\} \rightarrow \mathcal{D}_\dagger$ and $\Delta^m \times \{x\} \rightarrow \mathcal{D}$ each land in the maximal Kan complex contained in \mathcal{D}_\dagger and \mathcal{D} , respectively. Since $\iota\mathcal{D}_\dagger = \iota\mathcal{D}$, it now follows that one may find a lift $\Delta^m \times \mathcal{C}_\dagger \rightarrow \mathcal{D}_\dagger$ such that for any vertex $x \in \mathcal{C}_0$, the maps $\Delta^m \times \{x\} \rightarrow \mathcal{D}_\dagger$ land in the maximal Kan complex contained in \mathcal{D}_\dagger . \square

It will be convenient to describe pairs as certain functors of ∞ -categories. This will allow us to transfer pair structures back and forth along equivalences of ∞ -categories, and it will permit us to exhibit \mathbf{Pair}_∞ as a full subcategory of the arrow category $\mathcal{O}(\mathbf{Cat}_\infty)$.

1.14. **Definition.** Suppose C and D ∞ -categories. We say that a functor $D \rightarrow C$ *exhibits a pair structure on C* if it factors as an equivalence $D \xrightarrow{\sim} E$ followed by an inclusion $E \hookrightarrow C$ of a subcategory such that (C, E) is a pair.

1.15. Lemma. *Suppose C and D ∞ -categories. Then a functor $\psi: D \rightarrow C$ exhibits a pair structure on C if and only if the following conditions are satisfied.*

(1.15.1) *The functor ψ induces an equivalence $\iota D \rightarrow \iota C$.*

(1.15.2) *The functor ψ is a monomorphism in the ∞ -category \mathbf{Cat}_∞ ; i.e., the diagonal morphism*

$$D \rightarrow D \times_C^h D$$

in $h\mathbf{Cat}_\infty$ is an isomorphism.

Proof. Clearly any equivalence satisfies these criteria. If ψ is an inclusion of a subcategory such that (C, D) is a pair, then ψ , restricted to ιD , is the identity map, and it is an inner fibration such that the diagonal map $D \rightarrow D \times_C^h D$ is an isomorphism. This shows that if ψ exhibits a pair structure on C , then ψ satisfies the conditions listed.

Conversely, suppose ψ satisfies the conditions listed. Then it is hard not to show that for any objects $x, y \in D$, the functor ψ induces a homotopy monomorphism

$$\mathrm{Map}_D(x, y) \rightarrow \mathrm{Map}_C(\psi(x), \psi(y)),$$

whence the natural map

$$\mathrm{Map}_D(x, y) \rightarrow \mathrm{Map}_{NhD}(x, y) \times_{\mathrm{Map}_{NhC}(\psi(x), \psi(y))}^h \mathrm{Map}_C(\psi(x), \psi(y))$$

of $h\mathbf{Kan}$ is an isomorphism. This, combined with the fact that the map $\iota D \rightarrow \iota C$ is an equivalence, now implies that the natural map $D \rightarrow NhD \times_{NhC}^h C$ of $h\mathbf{Cat}_\infty$ is an isomorphism.

Since isomorphisms in hC are precisely equivalences in C , the induced functor $hD \rightarrow hC$ identifies hD with a subcategory of hC that contains all the isomorphisms. Denote by $hE \subset hC$ this subcategory. Now let E be the subcategory of C whose edges are those edges that map to $NhE \subset NhC$ under the canonical map $C \rightarrow NhC$; we thus have a diagram of ∞ -categories

$$\begin{array}{ccccc} D & \longrightarrow & E & \hookrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ NhD & \xrightarrow{\sim} & NhE & \hookrightarrow & NhC \end{array}$$

in which the square on the right and the big rectangle are homotopy pullbacks (for the Joyal model structure). Thus the square on the left is a homotopy pullback as well, and so the functor $D \rightarrow E$ is an equivalence, giving our desired factorization. \square

1.16. Proposition. *The functor $\mathbf{Pair}_\infty \rightarrow \mathcal{O}(\mathbf{Cat}_\infty)$ that sends a pair $(\mathcal{C}, \mathcal{C}_\dagger)$ to $\mathcal{C}_\dagger \rightarrow \mathcal{C}$ induces an equivalence between \mathbf{Pair}_∞ and the full subcategory of $\mathcal{O}(\mathbf{Cat}_\infty)$ spanned by those functors $D \rightarrow C$ that exhibit a pair structure on C .*

Proof. The essential surjectivity follows from Lm. 1.15, and the full faithfulness follows directly from the definitions of the mapping spaces in $\mathbf{Pair}_\infty^\Delta$. \square

It will be convenient for us to have a description of \mathbf{Pair}_∞ as a relative nerve. First, we record the following trivial result.

1.17. Lemma. *The following are equivalent for a functor of pairs $\psi: \mathcal{C} \rightarrow \mathcal{D}$.*

(1.17.1) *The functor of pairs ψ is an equivalence in the ∞ -category \mathbf{Pair}_∞ .*

(1.17.2) *The underlying functor of ∞ -categories is a categorical equivalence, and ψ is strict.*

(1.17.3) *The underlying functor of ∞ -categories is a categorical equivalence that induces an equivalence $h\mathcal{C}_\dagger \simeq h\mathcal{D}_\dagger$.*

This lemma, combined with Pr. 1.16 and 1.9, instantly yields the following.

1.18. Proposition. *Denote by $w\mathbf{Pair}_\infty^0 \subset \mathbf{Pair}_\infty^0$ the subcategory consisting of those functors of pairs $\mathcal{C} \rightarrow \mathcal{D}$ whose underlying functor of ∞ -categories is a categorical equivalence that induces an equivalence $h\mathcal{C}_\dagger \simeq h\mathcal{D}_\dagger$. Then \mathbf{Pair}_∞ is a relative nerve of $(\mathbf{Pair}_\infty^0, w\mathbf{Pair}_\infty^0)$.*

For use much later, it is helpful to have available the dual picture of categories with *fibrations*.

1.19. **Definition.** Suppose $(\mathcal{C}^{\text{op}}, (\mathcal{C}^{\text{op}})_{\dagger})$ a pair. Then write \mathcal{C}^{\dagger} for the subcategory $((\mathcal{C}^{\text{op}})_{\dagger})^{\text{op}} \subset \mathcal{C}$, and we call the morphisms of \mathcal{C}^{\dagger} *egressive* morphisms or *fibrations*. The pair $(\mathcal{C}, \mathcal{C}^{\dagger})$ will be called the *opposite pair* to $(\mathcal{C}^{\text{op}}, (\mathcal{C}^{\text{op}})_{\dagger})$. We will sometimes abuse terminology slightly by referring to $(\mathcal{C}, \mathcal{C}^{\dagger})$ as a *pair structure* on \mathcal{C}^{op} .

1.20. **Notation.** Suppose $(\mathcal{C}^{\text{op}}, (\mathcal{C}^{\text{op}})_{\dagger})$ a pair. Then a fibration of \mathcal{C} will frequently be denoted by a double headed arrow: \rightrightarrows . We will often abuse notation by simply writing \mathcal{C} for the opposite pair $(\mathcal{C}, \mathcal{C}^{\dagger})$.

1.21. **Lemma.** The formation $(\mathcal{C}, \mathcal{C}_{\dagger}) \mapsto (\mathcal{C}^{\text{op}}, (\mathcal{C}^{\text{op}})^{\dagger})$ of the opposite pair defines an involution $(-)^{\text{op}}$ of the ∞ -category \mathbf{Pair}_{∞} .

1.22. **Example.** Any ∞ -category C can be given the structure of a pair in two ways: the *minimal pair* $C^{\flat} := (C, \iota C)$ and the *maximal pair* $C^{\sharp} := (C, C)$. These specify a string of simplicial adjoints

$$\flat \dashv u \dashv \sharp \dashv \dagger$$

between $\mathbf{Cat}_{\infty}^{\Delta}$ and $\mathbf{Pair}_{\infty}^{\Delta}$, where u denotes the functor $(\mathcal{C}, \mathcal{C}_{\dagger}) \mapsto \mathcal{C}$, and \dagger denotes the functor $(\mathcal{C}, \mathcal{C}_{\dagger}) \mapsto \mathcal{C}_{\dagger}$.

1.23. **Notation.** For any space X , write

$$\mathcal{O}(X) := \text{Map}(\Delta^1, X).$$

If C is an ∞ -category, then $\mathcal{O}(C) = \text{Fun}(\Delta^1, C)$ is an ∞ -category as well [37, Pr. 1.2.7.3]; this is the *arrow ∞ -category* of C . (In fact, \mathcal{O} is a right Quillen functor for the Joyal model structure, since this model structure is cartesian.)

By [37, Cor. 2.4.7.11], the morphism $\mathcal{O}(C) \rightarrow C \times C$ induced by the inclusion

$$\Delta^{\{0\}} \sqcup \Delta^{\{1\}} \hookrightarrow \Delta^1$$

is a bifibration; hence evaluation at 0 defines a cartesian fibration $s: \mathcal{O}(C) \rightarrow C$, and evaluation at 1 defines a cocartesian fibration $t: \mathcal{O}(C) \rightarrow C$.

1.24. **Example.** Given a pair $\mathcal{C} = (\mathcal{C}, \mathcal{C}_{\dagger})$, we may give a pair structure to the ∞ -category $\mathcal{O}(\mathcal{C})$ in the following manner: define $\mathcal{O}_{\dagger}(\mathcal{C})$ as the pullback

$$\begin{array}{ccc} \mathcal{O}_{\dagger}(\mathcal{C}) & \longrightarrow & \mathcal{O}(\mathcal{C}) \\ \downarrow & & \downarrow s \\ \mathcal{C}_{\dagger} & \longrightarrow & \mathcal{C} \end{array}$$

1.25. **Notation.** For any simplicial set K , one has [37, Nt. 1.2.8.4] the right cone K^{\triangleright} and the left cone K^{\triangleleft} ; we write $+\infty$ for the cone point of K^{\triangleright} , and we write $-\infty$ for the cone point of K^{\triangleleft} .

1.26. **Example.** Denote by $\Lambda_0 \mathcal{Q}^2$ the pair $(\Lambda_0^2, \Delta^{\{0,1\}} \sqcup \Delta^{\{2\}})$:

$$\begin{array}{ccc} 0 & \xrightarrow{\quad} & 1 \\ \downarrow & & \\ 2 & & \end{array}$$

Denote by \mathcal{Q}^2 the pair $((\Lambda_0^2)^{\triangleright}, \Delta^{\{0,1\}} \sqcup \Delta^{\{2,\infty\}}) \cong (\Delta^1 \times \Delta^1, (\Delta^{\{0\}} \sqcup \Delta^{\{1\}}) \times \Delta^1)$:

$$\begin{array}{ccc} 0 & \xrightarrow{\quad} & 1 \\ \downarrow & & \downarrow \\ 2 & \xrightarrow{\quad} & \infty \end{array}$$

There is an obvious strict inclusion of pairs $\Lambda_0 \mathcal{Q}^2 \hookrightarrow \mathcal{Q}^2$.

2. WALDHAUSEN ∞ -CATEGORIES

The basic input for algebraic K -theory is a *Waldhausen ∞ -category*, which is a particular kind of pair. We introduce this concept here. First, however, we recall some basic facts about limits and colimits in ∞ -categories.

2.1. Notation. For any ∞ -category A and any ∞ -category C , we denote by

$$\mathrm{Colim}(A^\triangleright, C) \subset \mathrm{Fun}(A^\triangleright, C)$$

the full subcategory spanned by colimit diagrams $A^\triangleright \rightarrow C$.

2.2. A key result of Joyal [37, Pr. 1.2.12.9] states that for any functor $\psi: A \rightarrow C$, the fiber of the canonical restriction functor $\mathrm{Colim}(A^\triangleright, C) \rightarrow \mathrm{Fun}(A, C)$ over ψ is either empty or a contractible Kan space. One says that C *admits all A -shaped colimits* if the fibers of the functor

$$\mathrm{Colim}(A^\triangleright, C) \rightarrow \mathrm{Fun}(A, C)$$

are all nonempty. In this case, the functor $\mathrm{Colim}(A^\triangleright, C) \rightarrow \mathrm{Fun}(A, C)$ is an equivalence of ∞ -categories.

More generally, if \mathcal{A} is a family of ∞ -categories, then one says that C *admits all \mathcal{A} -shaped colimits* if the fibers of the functor $\mathrm{Colim}(A^\triangleright, C) \rightarrow \mathrm{Fun}(A, C)$ are all nonempty for every $A \in \mathcal{A}$.

Finally, if \mathcal{A} is a family of ∞ -categories, then a functor $f: C' \rightarrow C$ will be said to *preserve all \mathcal{A} -shaped colimits* if for any element $A \in \mathcal{A}$, the composite

$$\mathrm{Colim}(A^\triangleright, C') \rightarrow \mathrm{Fun}(A^\triangleright, C') \rightarrow \mathrm{Fun}(A^\triangleright, C)$$

factors through $\mathrm{Colim}(A^\triangleright, C) \subset \mathrm{Fun}(A^\triangleright, C)$.

2.3. Definition. A *zero object* is an object that is both initial and terminal.

The primary objects of study in this work are *Waldhausen ∞ -categories*.

2.4. Definition. A *Waldhausen ∞ -category* $(\mathcal{C}, \mathcal{C}_\dagger)$ is a pair of essentially small ∞ -categories such that the following axioms hold.

(2.4.1) The ∞ -category \mathcal{C} contains a zero object.

(2.4.2) For any zero object 0 , any morphism $0 \rightarrow X$ is ingressive.

(2.4.3) The source functor $s: \mathcal{C}_\dagger(\mathcal{C}) \rightarrow \mathcal{C}_\dagger$ is a cocartesian fibration, and an edge $\eta \in \mathcal{C}_\dagger(C)$ is s -cocartesian only if $t(\eta)$ is ingressive.

Call a functor of pairs $\psi: \mathcal{C} \rightarrow \mathcal{D}$ between two Waldhausen ∞ -categories *exact* if it satisfies the following conditions.

(2.4.4) The underlying functor ψ carries zero objects of \mathcal{C} to zero objects of \mathcal{D} .

(2.4.5) In the diagram

$$\begin{array}{ccc} \mathcal{C}_\dagger(\mathcal{C}) & \xrightarrow{\mathcal{C}_\dagger(\psi)} & \mathcal{C}_\dagger(\mathcal{D}) \\ s_{\mathcal{C}} \downarrow & & \downarrow s_{\mathcal{D}} \\ \mathcal{C}_\dagger & \xrightarrow{\psi_\dagger} & \mathcal{D}_\dagger \end{array}$$

the functor $\mathcal{C}_\dagger(\psi)$ sends $s_{\mathcal{C}}$ -cocartesian edges to $s_{\mathcal{D}}$ -cocartesian edges.

A *Waldhausen subcategory* of a Waldhausen ∞ -category \mathcal{C} is a subpair $\mathcal{D} \subset \mathcal{C}$ such that \mathcal{D} is a Waldhausen ∞ -category, and the inclusion $\mathcal{D} \hookrightarrow \mathcal{C}$ is exact.

2.5. The conditions demanded of a Waldhausen ∞ -category $(\mathcal{C}, \mathcal{C}_\dagger)$ can be rephrased in the following manner: there is a zero object in \mathcal{C} that is initial in \mathcal{C}_\dagger , and pushouts of cofibrations exist and are cofibrations [37, Lm. 6.1.1.1]. This last point can again be rephrased as the condition that the morphism of pairs $\Lambda_0 \mathcal{Q}^2 \rightarrow \mathcal{Q}^2$ (1.26) induces an equivalence of ∞ -categories

$$\mathrm{Colim}(\mathcal{Q}^2, \mathcal{C}) \times_{\mathrm{Fun}(\mathcal{Q}^2, \mathcal{C})} \mathrm{Fun}_{\mathrm{Pair}_\infty}^b(\mathcal{Q}^2, \mathcal{C}) \rightarrow \mathrm{Fun}_{\mathrm{Pair}_\infty}^b(\Lambda_0 \mathcal{Q}^2, \mathcal{C}).$$

An exact functor $\mathcal{C} \rightarrow \mathcal{D}$ is now one that preserves cofibrations, zero objects, and pushouts along cofibrations.

2.6. Notation. (2.6.1) Suppose \mathcal{C} and \mathcal{D} two Waldhausen ∞ -categories. We denote by $\text{Fun}_{\mathbf{Pair}_\infty}^b(\mathcal{C}, \mathcal{D})$ the full subcategory of $\text{Fun}_{\mathbf{Pair}_\infty}^b(\mathcal{C}, \mathcal{D})$ spanned by the exact functors $\mathcal{C} \rightarrow \mathcal{D}$ of Waldhausen ∞ -categories.

(2.6.2) Define $\mathbf{Wald}_\infty^\Delta$ as the following simplicial subcategory of $\mathbf{Pair}_\infty^\Delta$. The objects of $\mathbf{Wald}_\infty^\Delta$ are small Waldhausen ∞ -categories, and for any Waldhausen ∞ -categories \mathcal{C} and \mathcal{D} , the morphism space $\mathbf{Wald}_\infty^\Delta(\mathcal{C}, \mathcal{D})$ is defined by the formula

$$\mathbf{Wald}_\infty^\Delta(\mathcal{C}, \mathcal{D}) := \iota \text{Fun}_{\mathbf{Wald}_\infty}(\mathcal{C}, \mathcal{D}),$$

or equivalently, $\mathbf{Wald}_\infty^\Delta(\mathcal{C}, \mathcal{D})$ is the union of the connected components of $\mathbf{Pair}_\infty^\Delta(\mathcal{C}, \mathcal{D})$ corresponding to the exact morphisms.

2.7. Lemma. *The subcategory $\mathbf{Wald}_\infty \subset \mathbf{Pair}_\infty$ is stable under equivalences.*

2.8. Proposition. *Denote by $w\mathbf{Wald}_\infty^0 \subset \mathbf{Wald}_\infty^0$ the subcategory consisting of those exact functors $\mathcal{C} \rightarrow \mathcal{D}$ whose underlying functor of ∞ -categories is a categorical equivalence that induces an equivalence $h\mathcal{C}_\dagger \simeq h\mathcal{D}_\dagger$. The ∞ -category \mathbf{Wald}_∞ is the relative nerve of $(\mathbf{Wald}_\infty^0, w\mathbf{Wald}_\infty^0)$.*

2.9. Example. Equipped with the minimal pair structure, an ∞ -category C is a Waldhausen ∞ -category if and only if C is a contractible Kan complex.

Equipped with the maximal pair structure, any ∞ -category C that admits a zero object and all finite colimits can be regarded as a Waldhausen ∞ -category \mathcal{C} .

Equivalences between Waldhausen ∞ -categories with a maximal pair structure are easy to detect, thanks to the following result, which we can regard as a strengthening of Waldhausen's approximation theorem. Essentially the same result appears in work of Cisinski [16, Th. 2.15] and Blumberg–Mandell [14, Th. 1.3].

2.10. Proposition (Approximation). *Suppose \mathcal{C} and \mathcal{D} two ∞ -categories that become Waldhausen ∞ -categories when equipped with the maximal pair structure. Then an exact functor $\psi: \mathcal{C} \rightarrow \mathcal{D}$ is an equivalence if and only if it induces an equivalence of homotopy categories $h\mathcal{C} \xrightarrow{\sim} h\mathcal{D}$.*

Proof. Since \mathcal{C} and \mathcal{D} admit all finite colimits and since ψ preserves them, it follows that ψ preserves the tensor product with any finite space [37, Cor. 4.4.4.9]. Thus for any positive integer n and any morphism $\eta: X \rightarrow Y$ of \mathcal{C} , the map

$$\pi_n(\text{Map}_{\mathcal{C}}(X, Y), \eta) \rightarrow \pi_n(\text{Map}_{\mathcal{D}}(\psi(X), \psi(Y)), \psi(\eta))$$

can be identified with the fiber of the bijection

$$\pi_0 \text{Map}_{\mathcal{C}}(X \otimes S^n, Y) \rightarrow \pi_0 \text{Map}_{\mathcal{D}}(\psi(X) \otimes S^n, \psi(Y))$$

over $[\psi(\eta)] \in \pi_0 \text{Map}_{\mathcal{D}}(\psi(X), \psi(Y))$. Since this is a bijection, ψ is fully faithful, hence an equivalence. \square

2.11. Example. If $(C, \text{cof } C)$ is an ordinary category with cofibrations in the sense of Waldhausen [57, §1.1], then the pair $(NC, N(\text{cof } C))$ is easily seen to be a Waldhausen ∞ -category. If $(C, \text{cof } C, wC)$ is a category with cofibrations and weak equivalences in the sense of Waldhausen [57, §1.2], then one may endow the relative nerve $N(C, wC)$ of (C, wC) with a pair structure by defining the subcategory $N(C, wC)_\dagger \subset N(C, wC)$ as the smallest subcategory containing the equivalences and the images of the edges in NC corresponding to cofibrations. In Pr. 9.21, we will show that if (C, wC) is a “partial model category” in which the weak equivalences and trivial cofibrations are part of a three-arrow calculus of fractions, then $N(C, wC)$ is in fact a Waldhausen ∞ -category with this pair structure.

Entirely dual to the theory of Waldhausen ∞ -categories is the theory of *coWaldhausen ∞ -categories*. We record the definition here; clearly any result or construction in the theory of Waldhausen ∞ -categories can be immediately dualized.

2.12. Definition. (2.12.1) A *coWaldhausen ∞ -category* $(\mathcal{C}, \mathcal{C}^\dagger)$ is an opposite pair $(\mathcal{C}, \mathcal{C}^\dagger)$ such that the opposite $(\mathcal{C}^{\text{op}}, (\mathcal{C}^{\text{op}})_\dagger)$ is a Waldhausen ∞ -category.

(2.12.2) A functor of pairs $\psi: \mathcal{C} \rightarrow \mathcal{D}$ between two coWaldhausen ∞ -categories is said to be *exact* if its opposite $\psi^{\text{op}}: \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}^{\text{op}}$ is exact.

- 2.13. **Notation.** (2.13.1) Suppose \mathcal{C} and \mathcal{D} two coWaldhausen ∞ -categories. Denote by $\text{Fun}_{\mathbf{coWald}_\infty}(\mathcal{C}, \mathcal{D})$ the full subcategory of $\text{Fun}_{\mathbf{Pair}_\infty}(\mathcal{C}, \mathcal{D})$ spanned by the exact morphisms of coWaldhausen ∞ -categories.
- (2.13.2) Define $\mathbf{coWald}_\infty^\Delta$ as the following large simplicial subcategory of $\mathbf{Pair}_\infty^\Delta$. The objects of $\mathbf{coWald}_\infty^\Delta$ are small coWaldhausen ∞ -categories, and for any coWaldhausen ∞ -categories \mathcal{C} and \mathcal{D} , the morphism space $\mathbf{coWald}_\infty^\Delta(\mathcal{C}, \mathcal{D})$ is defined by the formula

$$\mathbf{coWald}_\infty^\Delta(\mathcal{C}, \mathcal{D}) := \iota \text{Fun}_{\mathbf{coWald}_\infty}(\mathcal{C}, \mathcal{D}),$$

or equivalently, $\mathbf{coWald}_\infty^\Delta(\mathcal{C}, \mathcal{D})$ is the union of the connected components of $\mathbf{Pair}_\infty^\Delta(\mathcal{C}, \mathcal{D})$ corresponding to the exact morphisms.

2.14. **Lemma.** *The opposite involution on \mathbf{Pair}_∞ restricts to an equivalence between \mathbf{Wald}_∞ and \mathbf{coWald}_∞ .*

3. WALDHAUSEN FIBRATIONS

Cartesian and cocartesian fibrations are well adapted to the study of “weak diagrams of ∞ -categories.” Similarly, we have a theory of *Waldhausen cartesian* and *cocartesian fibrations*, which make available a robust notion of “weak diagrams of Waldhausen ∞ -categories.” In order to introduce this idea, we first discuss *pair cartesian* and *cocartesian fibrations* in some detail.

3.1. **Definition.** Suppose S an ∞ -category. Then a *pair cartesian fibration* $\mathcal{X} \rightarrow S$ is a pair \mathcal{X} and a morphism of pairs $p: \mathcal{X} \rightarrow S^b$ such that the following conditions are satisfied.

(3.1.1) The underlying functor of p is a cartesian fibration.

(3.1.2) For any edge $\eta: s \rightarrow t$ of S , the induced functor $\eta^*: \mathcal{X}_t \rightarrow \mathcal{X}_s$ carries cofibrations to cofibrations.

A *pair cocartesian fibration* $\mathcal{X} \rightarrow S$ is a pair \mathcal{X} and a morphism of pairs $p: \mathcal{X} \rightarrow S^b$ such that $p^{\text{op}}: \mathcal{X}^{\text{op}} \rightarrow S^{\text{op}}$ is a pair cartesian fibration.

3.2. **Proposition.** *If S is an ∞ -category and $p: \mathcal{X} \rightarrow S$ is a pair cartesian fibration [respectively, a pair cocartesian fibration] with small fibers, then the functor $S^{\text{op}} \rightarrow \mathbf{Cat}_\infty$ [resp., the functor $S \rightarrow \mathbf{Cat}_\infty$] classified by p lifts to an essentially unique functor $S^{\text{op}} \rightarrow \mathbf{Pair}_\infty$ [resp., $S \rightarrow \mathbf{Pair}_\infty$].*

3.2.1. **Corollary.** *The classes of pair cartesian fibrations and pair cocartesian fibrations are each stable under base change.*

3.3. **Notation.** Denote by $\mathbf{Pair}_\infty^{\text{cart}}$ (respectively, $\mathbf{Pair}_\infty^{\text{cocart}}$) the following subcategory of $\text{Fun}(\Delta^1, \mathbf{Pair}_\infty)$. The objects of $\mathbf{Pair}_\infty^{\text{cart}}$ (resp., $\mathbf{Pair}_\infty^{\text{cocart}}$) are pair cartesian fibrations (resp., pair cocartesian fibrations) $\mathcal{X} \rightarrow S^b$. A commutative square

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\psi} & \mathcal{Y} \\ p \downarrow & & \downarrow q \\ S^b & \longrightarrow & T^b \end{array}$$

is a morphism of $\mathbf{Pair}_\infty^{\text{cart}}$ (resp., $\mathbf{Pair}_\infty^{\text{cocart}}$) if and only if ψ carries p -cartesian (resp. p -cocartesian) edges to q -cartesian (resp. q -cocartesian) edges.

By an abuse of notation, we will denote by (\mathcal{X}/S) an object $\mathcal{X} \rightarrow S$ of $\mathbf{Pair}_\infty^{\text{cart}}$ (resp., $\mathbf{Pair}_\infty^{\text{cocart}}$).

The following is immediate from Cor. 3.2.1 and [37, Lm. 6.1.1.1].

3.4. **Lemma.** *The target functors*

$$\mathbf{Pair}_\infty^{\text{cart}} \rightarrow \mathbf{Cat}_\infty \quad \text{and} \quad \mathbf{Pair}_\infty^{\text{cocart}} \rightarrow \mathbf{Cat}_\infty$$

induced by the inclusion $\{1\} \subset \Delta^1$ are both cartesian fibrations.

3.5. **Notation.** The fibers of the cartesian fibrations

$$\mathbf{Pair}_\infty^{\text{cart}} \rightarrow \mathbf{Cat}_\infty \quad \text{and} \quad \mathbf{Pair}_\infty^{\text{cocart}} \rightarrow \mathbf{Cat}_\infty$$

over an object $\{S\} \subset \mathbf{Cat}_\infty$ will be denoted $\mathbf{Pair}_{\infty/S}^{\text{cart}}$ and $\mathbf{Pair}_{\infty/S}^{\text{cocart}}$, respectively.

By an abuse of notation, denote by $\mathbf{Pair}_{\infty/S}^{\text{cart},0}$ [resp., by $\mathbf{Pair}_{\infty/S}^{\text{cocart},0}$] the subcategory of $\mathbf{Pair}_{\infty/S}$ whose objects are pair cartesian fibrations [resp., pair cocartesian fibrations] $\mathcal{X} \rightarrow S$ and whose morphisms are functors of pairs $\mathcal{X} \rightarrow \mathcal{Y}$ over S that carry cartesian morphisms to cartesian morphisms [resp., that carry cocartesian morphisms to cocartesian morphisms]. Denote by

$$w\mathbf{Pair}_{\infty/S}^{\text{cart},0} \subset \mathbf{Pair}_{\infty/S}^{\text{cart},0} \quad [\text{resp., by } w\mathbf{Pair}_{\infty/S}^{\text{cocart},0} \subset \mathbf{Pair}_{\infty/S}^{\text{cocart},0}]$$

the subcategory consisting of those morphisms $\mathcal{X} \rightarrow \mathcal{Y}$ such that for any vertex $s \in S_0$, the induced functor $\mathcal{X}_s \rightarrow \mathcal{Y}_s$ is a weak equivalence of pairs.

3.6. Lemma. *For any ∞ -category S , the ∞ -category $\mathbf{Pair}_{\infty/S}^{\text{cart}}$ [respectively, the ∞ -category $\mathbf{Pair}_{\infty/S}^{\text{cocart}}$] is a relative nerve of $(\mathbf{Pair}_{\infty/S}^{\text{cart},0}, w\mathbf{Pair}_{\infty/S}^{\text{cart},0})$ [resp., of $(\mathbf{Pair}_{\infty/S}^{\text{cocart},0}, w\mathbf{Pair}_{\infty/S}^{\text{cocart},0})$].*

Proof. To show that $\mathbf{Pair}_{\infty/S}^{\text{cart}}$ is a relative nerve of $(\mathbf{Pair}_{\infty/S}^{\text{cart},0}, w\mathbf{Pair}_{\infty/S}^{\text{cart},0})$, we first note that the analogous result for ∞ -categories of cartesian fibrations $X \rightarrow S$ holds. More precisely, let $\mathbf{Cat}_{\infty/S}^{\text{cart}}$ be the fiber of the target functor $t: \mathbf{Cat}_{\infty}^{\text{cart}} \rightarrow \mathbf{Cat}_{\infty}$ over $\{S\} \subset \mathbf{Cat}_{\infty}$, where $\mathbf{Cat}_{\infty}^{\text{cart}}$ denotes the subcategory of $\mathcal{O}(\mathbf{Cat}_{\infty})$ whose objects are cartesian fibration and whose morphisms carry cartesian morphisms to cartesian morphisms. Then $\mathbf{Cat}_{\infty/S}^{\text{cart}}$ may be identified with the nerve of the cartesian simplicial model category of marked simplicial sets over S [37, Pr. 3.1.3.7], whence it is the relative nerve of the category of cartesian fibrations over S , equipped with the cartesian equivalences.

To extend this result to a characterization of $\mathbf{Pair}_{\infty/S}^{\text{cart}}$ as a relative nerve, it suffices to note that $\mathbf{Pair}_{\infty/S}^{\text{cart}}$ is the full subcategory of the fiber product $\mathbf{Cat}_{\infty/S}^{\text{cart}} \times_{\mathbf{Cat}_{\infty}} \mathbf{Pair}_{\infty}$ spanned by the pair cartesian fibrations. \square

We may now employ this lemma to lift the equivalence $\mathbf{Cat}_{\infty/S}^{\text{cart}} \simeq \text{Fun}(S^{\text{op}}, \mathbf{Cat}_{\infty})$ of [37, §3.2] to an equivalence $\mathbf{Pair}_{\infty/S}^{\text{cart}} \simeq \text{Fun}(S^{\text{op}}, \mathbf{Pair}_{\infty})$.

3.7. Proposition. *For any ∞ -category S , the ∞ -category $\text{Fun}(S^{\text{op}}, \mathbf{Pair}_{\infty})$ [respectively, the ∞ -category $\text{Fun}(S, \mathbf{Pair}_{\infty})$] is a relative nerve of $(\mathbf{Pair}_{\infty/S}^{\text{cart},0}, w\mathbf{Pair}_{\infty/S}^{\text{cart},0})$ [resp., of $(\mathbf{Pair}_{\infty/S}^{\text{cocart},0}, w\mathbf{Pair}_{\infty/S}^{\text{cocart},0})$].*

Proof. The unstraightening functor of [37, §3.2] is a weak equivalence-preserving functor

$$\text{Un}^+: (\mathbf{Cat}_{\infty}^{\Delta})^{\mathcal{C}[S]^{\text{op}}} \rightarrow \mathbf{Cat}_{\infty/S}^{\text{cart}}$$

that induces an equivalence of relative nerves. For any simplicial functor $\mathcal{X}: \mathcal{C}[S]^{\text{op}} \rightarrow \mathbf{Pair}_{\infty}^{\Delta}$, then endow the unstraightening $\text{Un}^+(\mathcal{X})$ with a pair structure by letting $\text{Un}^+(\mathcal{X})_{\dagger} \subset \text{Un}^+(\mathcal{X})$ be the smallest subcategory containing all the equivalences as well as any cofibration of the of any fiber $\text{Un}^+(\mathcal{X})_s \cong \mathcal{X}(s)$. With this definition, we obtain a weak equivalence-preserving functor

$$\text{Un}^+: (\mathbf{Pair}_{\infty}^{\Delta})^{\mathcal{C}[S]^{\text{op}}} \rightarrow \mathbf{Pair}_{\infty/S}^{\text{cart},0}.$$

This functor induces a functor on relative nerves, which is essentially surjective by 3.2. Moreover, for any simplicial functors $\mathcal{X}, \mathcal{Y}: \mathcal{C}[S]^{\text{op}} \rightarrow \mathbf{Pair}_{\infty}^{\Delta}$, the simplicial set $\text{Map}_{N((\mathbf{Pair}_{\infty}^{\Delta})^{\mathcal{C}[S]^{\text{op}}})}(\mathcal{X}, \mathcal{Y})$ may be identified with the subspace of $\text{Map}_{N((\mathbf{Cat}_{\infty}^{\Delta})^{\mathcal{C}[S]^{\text{op}}})}(\mathcal{X}, \mathcal{Y})$ consisting of the connected components corresponding to natural transformations $\mathcal{X} \rightarrow \mathcal{Y}$ such that for any $s \in S_0$, the functor $\mathcal{X}_s \rightarrow \mathcal{Y}_s$ is a functor of pairs. Similarly, the simplicial set $\text{Map}_{\mathbf{Pair}_{\infty/S}^{\text{cart}}}(\text{Un}^+(\mathcal{X}), \text{Un}^+(\mathcal{Y}))$ may be identified with the subspace of $\text{Map}_{\mathbf{Cat}_{\infty/S}^{\text{cart}}}(\text{Un}^+(\mathcal{X}), \text{Un}^+(\mathcal{Y}))$ consisting of the connected components corresponding to functors $\text{Un}^+(\mathcal{X}) \rightarrow \text{Un}^+(\mathcal{Y})$ over S that send cartesian edges to cartesian edges with the additional property that for any $s \in S_0$, the functor

$$\text{Un}^+(\mathcal{X})_s \cong \mathcal{X}_s \rightarrow \mathcal{Y}_s \cong \text{Un}^+(\mathcal{Y})_s$$

is a functor of pairs. We thus conclude that Un^+ is fully faithful. \square

Armed with this, we may characterize colimits of pair cartesian fibrations fiberwise.

3.7.1. Corollary. Suppose S a small ∞ -category, K a small simplicial set. A functor $\mathcal{X}: K^\triangleright \rightarrow \mathbf{Pair}_{\infty/S}^{\text{cart}}$ [respectively, a functor $\mathcal{X}: K^\triangleright \rightarrow \mathbf{Pair}_{\infty/S}^{\text{cocart}}$] is a colimit diagram if and only if, for every vertex $s \in S_0$, the induced functor

$$\mathcal{X}_s: K^\triangleright \rightarrow \mathbf{Pair}_\infty$$

is a colimit diagram.

Of course the same characterization of limits holds, but it will not be needed.

The theory of pair cartesian and cocartesian fibrations is a relatively mild generalization of the theory of cartesian and cocartesian fibrations, and many of the results extend to this setting. In particular, we now set about proving a pair version of [37, Cor. 3.2.2.13].

3.8. Notation. Consider the ordinary category $s\mathbf{Set}(2)$ of pairs (V, U) consisting of a small simplicial set U and a simplicial subset $U \subset V$.

3.9. Proposition. Suppose $p: \mathcal{X} \rightarrow S$ a pair cartesian fibration, and suppose $q: \mathcal{Y} \rightarrow S$ a pair cocartesian fibration. Let $r: T_p \mathcal{Y} \rightarrow S$ be the map defined by the following universal property. We require, for any simplicial set K and any map $\sigma: K \rightarrow S$, a bijection

$$\text{Mor}_{/S}(K, T_p \mathcal{Y}) \cong \text{Mor}_{s\mathbf{Set}(2)/(S, \iota_S)}((K \times_S \mathcal{X}, K \times_S \mathcal{X}_\dagger), (\mathcal{Y}, \mathcal{Y}_\dagger)),$$

functorial in σ . Then r is a cocartesian fibration.

Proof. We may use [37, Cor. 3.2.2.13] to define a cocartesian fibration $r': T'_p \mathcal{Y} \rightarrow S$ with the universal property

$$\text{Mor}_{/S}(K, T'_p \mathcal{Y}) \cong \text{Mor}_{/S}(K \times_S \mathcal{X}, \mathcal{Y}).$$

Thus $T'_p \mathcal{Y}$ is an ∞ -category whose objects are pairs (s, ϕ) consisting of an object $s \in S_0$ and a functors $\phi: \mathcal{X}_s \rightarrow \mathcal{Y}_s$, and $T_p \mathcal{Y} \subset T'_p \mathcal{Y}$ is the full subcategory spanned by those pairs (s, ϕ) such that ϕ is a functor of pairs. An edge $(s, \phi) \rightarrow (t, \psi)$ in $T'_p \mathcal{Y}$ over an edge $\eta: s \rightarrow t$ of S is r' -cocartesian if and only if the corresponding natural transformation $\eta_{\mathcal{Y}, !} \circ \phi \circ \eta_{\mathcal{X}}^* \rightarrow \psi$ is an equivalence. Since composites of functors of pairs are again functors of pairs, it follows that if (s, ϕ) is an object of $T_p \mathcal{Y}$, then so is (t, ψ) , whence it follows that r is a cocartesian fibration. \square

Suppose S an ∞ -category, and suppose $p: \mathcal{X} \rightarrow S$ a cartesian fibration. The construction T_p is visibly a functor

$$\mathbf{Pair}_{\infty/S}^{\text{cocart}, 0} \rightarrow \mathbf{Pair}_{\infty/S}^{\text{cocart}, 0}.$$

To show that T_p defines a functor of ∞ -categories $\mathbf{Pair}_{\infty/S}^{\text{cocart}} \rightarrow \mathbf{Pair}_{\infty/S}^{\text{cocart}}$, it suffices by Lm. 3.6 just to observe that the functor T_p preserves the weak equivalences of $\mathbf{Pair}_{\infty/S}^{\text{cocart}, 0}$. Hence we have the following.

3.10. Proposition. Suppose $p: \mathcal{X} \rightarrow S$ a cartesian fibration; then the assignment $\mathcal{Y} \mapsto T_p \mathcal{Y}$ defines a functor

$$\mathbf{Pair}_{\infty/S}^{\text{cocart}} \rightarrow \mathbf{Pair}_{\infty/S}^{\text{cocart}}.$$

Now we have laid the groundwork for a theory of *Waldhausen cartesian* and *cocartesian* fibrations.

3.11. Definition. Suppose S an ∞ -category. A *Waldhausen cartesian fibration* $p: \mathcal{X} \rightarrow S$ is a pair cartesian (resp., cocartesian) fibration satisfying the following conditions.

(3.11.1) For any object s of S , the pair

$$\mathcal{X}_s := (\mathcal{X} \times_S \{s\}, \mathcal{X}_\dagger \times_S \{s\})$$

is a Waldhausen ∞ -category.

(3.11.2) For any morphism $\eta: s \rightarrow t$, the corresponding functor of pairs $\eta^*: \mathcal{X}_t \rightarrow \mathcal{X}_s$ is an exact functor of Waldhausen ∞ -categories.

Dually, a *Waldhausen cocartesian fibration* is a pair \mathcal{X} and a functor of pairs $p: \mathcal{X} \rightarrow S$ such that $p^{\text{op}}: \mathcal{X}^{\text{op}} \rightarrow S^{\text{op}}$ is a Waldhausen cartesian fibration.

As with pair cartesian fibrations, Waldhausen cartesian fibrations classify functors to \mathbf{Wald}_∞ :

3.12. Proposition. *Suppose S an ∞ -category. Then a pair cartesian [respectively, cocartesian] fibration $p: \mathcal{X} \rightarrow S$ is a Waldhausen cartesian fibration [resp., a Waldhausen cocartesian fibration] if and only if the functor $S^{\text{op}} \rightarrow \mathbf{Pair}_\infty$ [resp., the functor $S \rightarrow \mathbf{Pair}_\infty$] classified by p lifts to an essentially unique functor $S^{\text{op}} \rightarrow \mathbf{Wald}_\infty$ [resp., $S \rightarrow \mathbf{Wald}_\infty$].*

3.12.1. Corollary. *The classes of Waldhausen cartesian fibrations and Waldhausen cocartesian fibrations are each stable under base change.*

3.13. Notation. Denote by $\mathbf{Wald}_\infty^{\text{cart}}$ (respectively, $\mathbf{Wald}_\infty^{\text{cocart}}$) the following subcategory of $\mathbf{Pair}_\infty^{\text{cart}}$ (resp., $\mathbf{Pair}_\infty^{\text{cocart}}$). The objects of $\mathbf{Wald}_\infty^{\text{cart}}$ (resp., $\mathbf{Wald}_\infty^{\text{cocart}}$) are Waldhausen cartesian fibrations (resp., Waldhausen cocartesian fibrations) $\mathcal{X} \rightarrow S$ with S and \mathcal{X} small. A morphism

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\psi} & \mathcal{Y} \\ \downarrow & & \downarrow \\ S^{\flat} & \xrightarrow[\phi]{} & T^{\flat} \end{array}$$

of $\mathbf{Pair}_\infty^{\text{cart}}$ (resp., $\mathbf{Pair}_\infty^{\text{cocart}}$) is a morphism of $\mathbf{Wald}_\infty^{\text{cart}}$ (resp., $\mathbf{Wald}_\infty^{\text{cocart}}$) if and only if ψ induces exact functors $\mathcal{X}_s \rightarrow \mathcal{Y}_{\phi(s)}$ for every vertex $s \in S_0$.

The following is again a consequence of [37, Lm. 6.1.1.1].

3.14. Lemma. *The target functors*

$$\mathbf{Wald}_\infty^{\text{cart}} \rightarrow \mathbf{Cat}_\infty \quad \text{and} \quad \mathbf{Wald}_\infty^{\text{cocart}} \rightarrow \mathbf{Cat}_\infty$$

induced by the inclusion $\{1\} \subset \Delta^1$ are both cartesian fibrations.

3.15. Notation. The fibers of the cartesian fibrations

$$\mathbf{Wald}_\infty^{\text{cart}} \rightarrow \mathbf{Cat}_\infty \quad \text{and} \quad \mathbf{Wald}_\infty^{\text{cocart}} \rightarrow \mathbf{Cat}_\infty$$

over an object $\{S\} \subset \mathbf{Cat}_\infty$ will be denoted $\mathbf{Wald}_{\infty/S}^{\text{cart}}$ and $\mathbf{Wald}_{\infty/S}^{\text{cocart}}$, respectively.

The following proposition is now an easy extension of the argument given in the proof of Pr. 3.7.

3.16. Proposition. *The equivalence of ∞ -categories $\mathbf{Pair}_{\infty/S}^{\text{cart}} \simeq \text{Fun}(S^{\text{op}}, \mathbf{Pair}_\infty)$ [respectively, the equivalence of ∞ -categories $\mathbf{Pair}_{\infty/S}^{\text{cocart}} \simeq \text{Fun}(S, \mathbf{Pair}_\infty)$] of Pr. 3.7 restricts to an equivalence of ∞ -categories*

$$\mathbf{Wald}_{\infty/S}^{\text{cart}} \simeq \text{Fun}(S^{\text{op}}, \mathbf{Wald}_\infty) \quad [\text{resp.,} \quad \mathbf{Wald}_{\infty/S}^{\text{cocart}} \simeq \text{Fun}(S, \mathbf{Wald}_\infty) \quad].$$

As with pair fibrations, colimits of Waldhausen cartesian fibrations may be characterized fiberwise.

3.16.1. Corollary. *Suppose S a small ∞ -category, K a small simplicial set. A functor $\mathcal{X}: K^\triangleright \rightarrow \mathbf{Wald}_{\infty/S}^{\text{cart}}$ [respectively, a functor $\mathcal{X}: K^\triangleright \rightarrow \mathbf{Wald}_{\infty/S}^{\text{cocart}}$] is a colimit diagram if and only if, for every vertex $s \in S_0$, the induced functor*

$$\mathcal{X}_s: K^\triangleright \rightarrow \mathbf{Wald}_\infty$$

is a colimit diagram.

4. VIRTUAL WALDHAUSEN ∞ -CATEGORIES

We begin by constructing limits and some colimits in the ∞ -category \mathbf{Wald}_∞ . We do this first by analyzing limits and colimits in the ∞ -category \mathbf{Pair}_∞ .

4.1. Suppose C a locally small ∞ -category [37, Df. 5.4.1.3]. For a regular cardinal $\kappa < \kappa_0$, recall [37, Df. 5.5.7.1] that C is said to be κ -compactly generated (or simply *compactly generated* if $\kappa = \omega$) if it is κ -accessible and admits all small colimits. From this it will follow that C admits all small limits as well. It follows from Simpson's theorem [37, Th. 5.5.1.1] that C is κ -compactly generated if and only if it is a κ -accessible localization of the ∞ -category of presheaves $\mathcal{P}(C_0) = \text{Fun}(C_0^{\text{op}}, \mathbf{Kan})$ of small spaces on some small ∞ -category C_0 .

4.2. Proposition. *The ∞ -category \mathbf{Pair}_∞ is an ω -accessible localization of the arrow ∞ -category $\mathcal{O}(\mathbf{Cat}_\infty)$.*

Proof. We use 1.16 to identify \mathbf{Pair}_∞ with a full subcategory of $\mathcal{O}(\mathbf{Cat}_\infty)$. Now the condition that an object $D \rightarrow C$ of $\mathcal{O}(\mathbf{Cat}_\infty)$ be a monomorphism is equivalent to the demand that the functors

$$\iota D \rightarrow \iota D \times_{\iota C}^h \iota D \quad \text{and} \quad \iota \mathcal{O}(D) \rightarrow \iota \mathcal{O}(D) \times_{\iota \mathcal{O}(C)}^h \iota \mathcal{O}(D)$$

be isomorphisms of $h\mathbf{Cat}_\infty$. This, in turn, is the requirement that the object $D \rightarrow C$ be S -local, where S is the set

$$S := \left\{ \begin{array}{ccc} \Delta^p \sqcup \Delta^p & \xrightarrow{\nabla} & \Delta^p \\ \nabla \downarrow & & \parallel \\ \Delta^p & \xlongequal{\quad} & \Delta^p \end{array} \quad \middle| \quad \mathbf{p} \in \Delta \right\}$$

of morphisms of $\mathcal{O}(\mathbf{Cat}_\infty)$. The condition that an object $D \rightarrow C$ of $\mathcal{O}(\mathbf{Cat}_\infty)$ induce an equivalence $\iota D \rightarrow \iota C$ is equivalent to the requirement that it be local with respect to the singleton

$$\{\phi: (\Delta^0)^\flat \rightarrow (\Delta^0)^\sharp\}.$$

Hence \mathbf{Pair}_∞ is equivalent to the full subcategory of the $S \cup \{\phi\}$ -local objects of $\mathcal{O}(\mathbf{Cat}_\infty)$. Now it is easy to see that the $S \cup \{\phi\}$ -local objects of $\mathcal{O}(\mathbf{Cat}_\infty)$ are closed under filtered colimits; hence by [37, Pr. 5.5.3.6 and Cor. 5.5.7.3], the ∞ -category \mathbf{Pair}_∞ is an ω -accessible localization. \square

4.2.1. Corollary. *The ∞ -category \mathbf{Pair}_∞ is compactly generated.*

4.2.2. Corollary. *The ∞ -category \mathbf{Pair}_∞ admits all small limits, and the inclusion*

$$\mathbf{Pair}_\infty \hookrightarrow \mathcal{O}(\mathbf{Cat}_\infty)$$

preserves them.

4.2.3. Corollary. *The ∞ -category \mathbf{Pair}_∞ admits all small colimits, and the inclusion*

$$\mathbf{Pair}_\infty \hookrightarrow \mathcal{O}(\mathbf{Cat}_\infty)$$

preserves small filtered colimits.

4.2.4. Corollary. *Any pair \mathcal{C} is the colimit of its compact subpairs.*

Now we construct limits in \mathbf{Wald}_∞ .

4.3. Proposition. *The ∞ -category \mathbf{Wald}_∞ admits all small limits, and the inclusion functor $\mathbf{Wald}_\infty \rightarrow \mathbf{Pair}_\infty$ preserves them.*

Proof. We employ [37, Pr. 4.4.2.6] to reduce the problem to proving the existence of products and pullbacks in \mathbf{Wald}_∞ . To complete the proof, we make the following observations.

(4.3.1) Suppose I a set, suppose $(\mathcal{C}_i)_{i \in I}$ an I -tuple of pairs of ∞ -categories, and suppose \mathcal{C} the product of these pairs. If for each $i \in I$, the pair \mathcal{C}_i is a Waldhausen ∞ -category, then so is \mathcal{C} . Moreover, if \mathcal{D} is a Waldhausen ∞ -category, then a functor of pairs $\mathcal{D} \rightarrow \mathcal{C}$ is exact if and only if the composite

$$\mathcal{D} \rightarrow \mathcal{C} \rightarrow \mathcal{C}_i$$

is exact for any $i \in I$. This follows directly from the fact that limits and colimits of a product are computed objectwise [37, Cor. 5.1.2.3].

(4.3.2) Suppose

$$\begin{array}{ccc} \mathcal{C}' & \xrightarrow{q'} & \mathcal{F}' \\ p' \downarrow & & \downarrow p \\ \mathcal{C} & \xrightarrow{q} & \mathcal{F} \end{array}$$

a pullback diagram of pairs of ∞ -categories. Suppose moreover that \mathcal{C} , \mathcal{F} , and \mathcal{F}' are all Waldhausen ∞ -categories, and p and q are exact functors. Then \mathcal{C}' is a Waldhausen ∞ -category, and for any Waldhausen ∞ -category \mathcal{D} , a functor of pairs $\psi: \mathcal{D} \rightarrow \mathcal{C}$ is exact if and only if the composites $p' \circ \psi$ and $q' \circ \psi$ are exact. This follows from [37, Lm. 5.4.5.5]. \square

We obtain a similar characterization of filtered colimits in \mathbf{Wald}_∞ .

4.4. Proposition. *The ∞ -category \mathbf{Wald}_∞ admits all small filtered colimits, and the inclusion functor $\mathbf{Wald}_\infty \rightarrow \mathbf{Pair}_\infty$ preserves them.*

Proof. Suppose A a filtered ∞ -category, and suppose $A \rightarrow \mathbf{Wald}_\infty$ a functor given by the assignment $a \mapsto \mathcal{C}_a$, and suppose \mathcal{C} the colimit of the composite functor

$$A \rightarrow \mathbf{Wald}_\infty \rightarrow \mathbf{Pair}_\infty$$

The proof is completed by the following observations.

(4.4.1) Pushouts of cofibrations in \mathcal{C} exist and are cofibrations. This follows by precisely the same argument as [37, Pr. 5.5.7.11].

(4.4.2) The underlying ∞ -category \mathcal{C} admits a zero object, which is the initial object of \mathcal{C}_\dagger . This follows from the fact that for any object $a \in A$, the functor $\mathcal{C}_a \rightarrow \mathcal{C}$ preserves zero objects, and the functor $\mathcal{C}_{a,\dagger} \rightarrow \mathcal{C}_\dagger$ preserves initial objects. \square

We now show that \mathbf{Wald}_∞ also admits finite *direct sums*, i.e., that finite products in \mathbf{Wald}_∞ are also finite coproducts.

4.5. Definition. Suppose C is an ∞ -category. Then C is said to *admit finite direct sums* if the following conditions hold.

(4.5.1) The ∞ -category C is pointed.

(4.5.2) The ∞ -category C has all finite products and coproducts.

(4.5.3) For any finite set I and any I -tuple $(X_i)_{i \in I}$ of objects of C , the map

$$\coprod X_I \rightarrow \prod X_I$$

in hC — given by the maps $\phi_{ij}: X_i \rightarrow X_j$, where ϕ_{ij} is zero unless $i = j$, in which case it is the identity — is an isomorphism.

If C admits finite direct sums, then for any finite set I and any I -tuple $(X_i)_{i \in I}$ of objects of C , we denote by $\bigoplus X_I$ the product (or, equivalently, the coproduct) of the X_i .

4.6. Proposition. *The ∞ -category \mathbf{Wald}_∞ admits finite direct sums.*

Proof. The Waldhausen ∞ -category Δ^0 is a zero object. To complete the proof, it suffices to show that for any finite set I and any I -tuple of Waldhausen ∞ -categories $(\mathcal{C}_i)_{i \in I}$ with product \mathcal{C} , the functors $\phi_i: \mathcal{C}_i \rightarrow \mathcal{C}$ — given by the functors $\phi_{ij}: \mathcal{C}_i \rightarrow \mathcal{C}_j$, where ϕ_{ij} is zero unless $j = i$, in which case it is the identity — are exact and exhibit \mathcal{C} as the *coproduct* of $(\mathcal{C}_i)_{i \in I}$. To prove this, we employ [37, Th. 4.2.4.1] to reduce the problem to showing that for any Waldhausen ∞ -category \mathcal{D} , the map

$$\mathbf{Wald}_\infty^\Delta(\mathcal{C}, \mathcal{D}) \rightarrow \prod_{i \in I} \mathbf{Wald}_\infty^\Delta(\mathcal{C}_i, \mathcal{D})$$

induced by the functor ϕ_i is a weak homotopy equivalence. We prove the stronger claim that the functor

$$w: \mathbf{Fun} \mathbf{Wald}_\infty(\mathcal{C}, \mathcal{D}) \rightarrow \prod_{i \in I} \mathbf{Fun} \mathbf{Wald}_\infty(\mathcal{C}_i, \mathcal{D})$$

is an equivalence of ∞ -categories.

For this, consider the following composite

$$\prod_{i \in I} \mathbf{Fun}(\mathcal{C}_i, \mathcal{D}) \xrightarrow{u} \mathbf{Fun}(\mathcal{C}, \mathbf{Fun}(NI, \mathcal{D})) \xrightarrow{r} \mathbf{Fun}(\mathcal{C}, \mathbf{Colim}((NI)^\triangleright, \mathcal{D})) \xrightarrow{e} \mathbf{Fun}(\mathcal{C}, \mathcal{D})$$

where u is the functor corresponding to the functor

$$\mathcal{C} \times \prod_{i \in I} \mathbf{Fun}(\mathcal{C}_i, \mathcal{D}) \cong \prod_{i \in I} (\mathcal{C}_i \times \mathbf{Fun}(\mathcal{C}_i, \mathcal{D})) \rightarrow \prod_{i \in I} \mathcal{D},$$

where r is a section of the trivial fibration

$$\mathbf{Fun}(\mathcal{C}, \mathbf{Colim}((NI)^\triangleright, \mathcal{D})) \rightarrow \mathbf{Fun}(\mathcal{C}, \mathbf{Fun}(NI, \mathcal{D})),$$

and e is the functor induced by the functor $\text{Colim}((NI)^\triangleright, \mathcal{D}) \rightarrow \mathcal{D}$ given by evaluation at the cone point ∞ . This composite restricts to a functor

$$v: \prod_{i \in I} \text{Fun}_{\mathbf{Wald}_\infty}(\mathcal{C}_i, \mathcal{D}) \rightarrow \text{Fun}_{\mathbf{Wald}_\infty}(\mathcal{C}, \mathcal{D});$$

indeed, one checks directly that if $(\psi_i: \mathcal{C}_i \rightarrow \mathcal{D})_{i \in I}$ is an I -tuple of exact functors, then a functor $\psi: \mathcal{C} \rightarrow \mathcal{D}$ that sends a simplex $\sigma = (\sigma_i)_{i \in I}$ to a coproduct $\coprod_{i \in I} \psi_i(\sigma_i)$ in \mathcal{D} is exact, and the situation is similar for natural transformations of exact functors.

We claim that the functor v is a homotopy inverse to w . A homotopy $w \circ v \simeq \text{id}$ can be constructed directly from the canonical equivalences

$$Y \simeq Y \sqcup \coprod_{i \in I - \{j\}} 0_i$$

for any zero objects 0_i in \mathcal{D} . In the other direction, the existence of a homotopy $v \circ w \simeq \text{id}$ follows from the observation that the natural transformations $\phi_i \circ \text{pr}_i \rightarrow \text{id}$ exhibit the identity functor on \mathcal{C} as the coproduct $\coprod_{i \in I} \phi_i \circ \text{pr}_i$. \square

Since any small coproduct can be written as a filtered colimit of finite coproducts, we deduce the following.

4.6.1. Corollary. *The ∞ -category \mathbf{Wald}_∞ admits all small coproducts.*

Finally, we set about showing that \mathbf{Wald}_∞ is ω -accessible. In fact, we prove the following stronger result.

4.7. Proposition. *The ∞ -category \mathbf{Wald}_∞ is compactly generated.*

Proof. We wish to show that \mathbf{Wald}_∞ is an object of the (huge) ∞ -category \mathbf{Pr}_ω^R of large compactly generated ∞ -categories and limit-preserving, ω -continuous functors [37, Df. 5.5.7.5]. Thanks to [37, Pr. 5.5.7.6], it suffices for us to exhibit \mathbf{Wald}_∞ as a suitable limit.

Let S be the ∞ -category

$$(\Delta^1 \cup^{\partial \Delta^1} \Delta^1) \cup^{\Delta^{\{0\}}} (\Delta^1 \cup^{\partial \Delta^1} \Delta^1).$$

This ∞ -category is in fact the nerve of an ordinary category with three objects, o , z , and w , two maps $o \rightarrow z$, two maps $o \rightarrow w$, and no other nonidentity maps. We may now construct a functor $W: S \rightarrow \mathbf{Pr}_\omega^R$ in the following manner. Let W assign to o the ∞ -category \mathbf{Pair}_∞ , to z the ∞ -category \mathbf{Cat}_∞ , and to w the ∞ -category \mathbf{Cat}_∞ . The two maps $o \rightarrow z$ are sent to: (1) the constant functor at the object $\Delta^0 \in \mathbf{Cat}_\infty$, (2) the functor

$$\mathcal{C} \mapsto \text{Colim}(\Delta^0, \mathcal{C}_\dagger) \times_{\text{Fun}(\Delta^0, \mathcal{C})} \text{Colim}(\Delta^0, \mathcal{C}) \times_{\text{Fun}(\Delta^0, \mathcal{C})} \text{Lim}(\Delta^0, \mathcal{C}).$$

The two maps $o \rightarrow w$ are sent to: (1) the functor

$$\mathcal{C} \mapsto \text{Fun}_{\mathbf{Pair}_\infty}^b(\Lambda_0 \mathcal{Q}^2, \mathcal{C})$$

and (2) the functor

$$\mathcal{C} \mapsto \text{Colim}(\mathcal{Q}^2, \mathcal{C}) \times_{\text{Fun}(\mathcal{Q}^2, \mathcal{C})} \text{Fun}_{\mathbf{Pair}_\infty}^b(\mathcal{Q}^2, \mathcal{C}).$$

It is not difficult to see that these functors are all limit-preserving and ω -continuous. Now we observe that \mathbf{Wald}_∞ may be exhibited as a limit of the functor

$$S \xrightarrow{W} \mathbf{Pr}_\omega^R \hookrightarrow \mathbf{Cat}_\infty(\kappa_1),$$

whence we deduce that \mathbf{Wald}_∞ is compactly generated from [37, Pr. 5.5.7.6]. \square

This result shows that in fact the ∞ -category \mathbf{Wald}_∞ admits *all* small colimits, not only the filtered ones. However, these other colimits are not preserved by the sorts of invariants in which we are interested, and so we will regard them as pathological. Nevertheless, we will have use for the following.

4.7.1. Corollary. *The ∞ -category \mathbf{Wald}_∞ is ω -accessible.*

4.7.2. Corollary. *The ∞ -category \mathbf{Wald}_∞ may be identified with the Ind-objects of the full subcategory $\mathbf{Wald}_\infty^\omega \subset \mathbf{Wald}_\infty$ spanned by the compact Waldhausen ∞ -categories.*

We obtain a further corollary by combining Prs. 4.7, 4.3, and 4.4 together with the adjoint functor theorem.

4.7.3. **Corollary.** *The forgetful functor $\mathbf{Wald}_\infty \rightarrow \mathbf{Pair}_\infty$ admits a left adjoint $W: \mathbf{Pair}_\infty \rightarrow \mathbf{Wald}_\infty$.*

4.8. Since the opposite functor $\mathbf{Wald}_\infty \rightarrow \mathbf{coWald}_\infty$ is an equivalence of ∞ -categories, it follows that the entire crop of structural results also hold for \mathbf{coWald}_∞ . That is, \mathbf{coWald}_∞ admits all small limits and all small filtered colimits, and the inclusion functor $\mathbf{coWald}_\infty \rightarrow \mathbf{Pair}_\infty$ preserves each of them. Similarly, \mathbf{coWald}_∞ admits finite direct sums and all small coproducts, and it is compactly generated.

Now we are prepared to introduce a convenient enlargement of the ∞ -category \mathbf{Wald}_∞ . In effect, we aim to “correct” the colimits of \mathbf{Wald}_∞ that we regard as pathological. To this we add formal geometric realizations and nothing more. The result is the ∞ -category whose homotopy theory forms the basis of our work here.

4.9. **Notation.** For any ∞ -category C , we shall write $\mathcal{P}(C)$ for the ∞ -category $\mathbf{Fun}(C^{\mathrm{op}}, \mathbf{Kan})$ of presheaves of small spaces on C . If C is locally small, then there exists a Yoneda embedding

$$j: C \hookrightarrow \mathcal{P}(C).$$

4.10. **Definition.** A *virtual Waldhausen ∞ -category* is a functor $\mathcal{T}: (\mathbf{Wald}_\infty^\omega)^{\mathrm{op}} \rightarrow \mathbf{Kan}$ that preserves products.

4.11. **Notation.** Denote by \mathbf{VWald}_∞ the full subcategory of $\mathcal{P}(\mathbf{Wald}_\infty^\omega)$ spanned by the virtual Waldhausen ∞ -categories. In other words, \mathbf{VWald}_∞ is nonabelian derived ∞ -category of $\mathbf{Wald}_\infty^\omega$ [37, §5.5.8].

4.12. In the notation of [37, Df. 5.3.6.5], the ∞ -category \mathbf{VWald}_∞ can be identified with any of the following ∞ -categories:

(4.12.1) the ∞ -category $\mathcal{P}_{\mathcal{D}}^{\mathcal{K}} \mathbf{Wald}_\infty^\omega$, where \mathcal{D} is the collection of finite discrete simplicial sets, and \mathcal{K} is the collection of small simplicial sets,

(4.12.2) the ∞ -category $\mathcal{P}_{\mathcal{G}}^{\mathcal{G}} \mathbf{Wald}_\infty^\omega$, where \mathcal{G} is the collection of small, sifted simplicial sets, and

(4.12.3) the ∞ -category $\mathcal{P}_{\mathcal{F}}^{\mathcal{F}} \mathbf{Wald}_\infty^\omega$, where \mathcal{F} is the collection of small, filtered simplicial sets.

Only the last of these characterizations requires an explanation, but it follows directly from 4.7.2 and the transitivity [37, Pr. 5.3.6.11].

We may summarize these characterizations by saying that the Yoneda embedding is a fully faithful functor

$$j: \mathbf{Wald}_\infty \hookrightarrow \mathbf{VWald}_\infty$$

that induces an equivalence

$$\mathbf{Fun}^L(\mathbf{VWald}_\infty, D) \xrightarrow{\sim} \mathbf{Fun}_{\mathcal{D}}(\mathbf{Wald}_\infty^\omega, D)$$

for any ∞ -category D that admits all small colimits and equivalences

$$\mathbf{Fun}_{\mathcal{G}}(\mathbf{VWald}_\infty, D') \xrightarrow{\sim} \mathbf{Fun}(\mathbf{Wald}_\infty^\omega, D') \quad \text{and} \quad \mathbf{Fun}_{\mathcal{F}}(\mathbf{VWald}_\infty, D') \xrightarrow{\sim} \mathbf{Fun}_{\mathcal{F}}(\mathbf{Wald}_\infty, D')$$

for any ∞ -category D' that admits small sifted colimits.

4.13. **Proposition.** *The ∞ -category \mathbf{VWald}_∞ is compactly generated. Moreover, it admits all direct sums, and the inclusion j preserves them.*

Proof. The first statement is [37, Pr. 5.5.8.10(8)]. To see that \mathbf{VWald}_∞ admits direct sums, we use the fact that we may exhibit any object of \mathbf{VWald}_∞ as a sifted colimit of compact Waldhausen ∞ -categories in $\mathcal{P}(\mathbf{Wald}_\infty^\omega)$ [37, Lm. 5.5.8.14]; now since sifted colimits commute with both finite products [37, Lm. 5.5.8.11] and coproducts, and since j preserves products and finite coproducts [37, Lm. 5.5.8.10(2)], the proof is complete. \square

4.14. **Definition.** Suppose D an ∞ -category that admits all sifted colimits. Then a functor

$$\Phi: \mathbf{VWald}_\infty \rightarrow D$$

that preserves all sifted colimits will be said to be the *left derived functor* of the corresponding ω -continuous functor $\phi = \Phi \circ j: \mathbf{Wald}_\infty \rightarrow D$ or of the further restriction $\mathbf{Wald}_\infty^\omega \rightarrow D$ of ϕ to $\mathbf{Wald}_\infty^\omega$.

We now give an explicit construction of colimits in \mathbf{VWald}_∞ of sifted diagrams of Waldhausen ∞ -categories when they are exhibited as Waldhausen cocartesian fibrations to a sifted ∞ -category.

4.15. Construction. Suppose S an ∞ -category, and suppose $\mathcal{X} \rightarrow S$ a Waldhausen cocartesian fibration. Then for any compact Waldhausen ∞ -category \mathcal{C} , define a simplicial set $\mathcal{H}'(\mathcal{C}, (\mathcal{X}/S))$ over S via the universal property

$$\mathrm{Mor}_S(K, \mathcal{H}'(\mathcal{C}, (\mathcal{X}/S))) \cong \mathrm{Mor}_S(\mathcal{C} \times K, \mathcal{X}).$$

The resulting morphism $p: \mathcal{H}'(\mathcal{C}, (\mathcal{X}/S)) \rightarrow S$ is a cocartesian fibration by [37, Cor. 3.2.2.13]. Denote by $\mathcal{H}(\mathcal{C}, (\mathcal{X}/S))$ the full subcategory of $\mathcal{H}'(\mathcal{C}, (\mathcal{X}/S))$ spanned by those functors $\mathcal{C} \rightarrow \mathcal{Y}_s$ that are exact functors of Waldhausen ∞ -categories; here too the canonical map to S can be shown to be a cocartesian fibration. Now denote by $\mathrm{H}(\mathcal{C}, (\mathcal{X}/S))$ the subcategory $\iota_S \mathcal{H}(\mathcal{C}, (\mathcal{X}/S))$ of $\mathcal{H}(\mathcal{C}, (\mathcal{X}/S))$ consisting of the p -cocartesian morphisms.

By [37, Cor. 3.3.4.6], for any compact Waldhausen ∞ -category \mathcal{C} and any Waldhausen cocartesian fibration $p: \mathcal{X} \rightarrow S$, the simplicial set $\mathrm{H}(\mathcal{C}, (\mathcal{X}/S))$ is weakly homotopy equivalent to the colimit of the composite

$$S \xrightarrow{\widetilde{\mathcal{X}}} \mathbf{Wald}_\infty \xrightarrow{j} \mathcal{P}(\mathbf{Wald}_\infty) \xrightarrow{\mathrm{ev}_{\mathcal{Y}}} \mathbf{Kan},$$

where $\widetilde{\mathcal{X}}$ is the functor classified by p , the embedding j in the Yoneda embedding, and $\mathrm{ev}_{\mathcal{Y}}$ is evaluation at the object \mathcal{Y} . That is, in $h\mathbf{Kan}$, we have

$$\mathrm{H}(\mathcal{C}, (\mathcal{X}/S)) \simeq \mathrm{hocolim}_{s \in S} \mathbf{Wald}_\infty^\Delta(\mathcal{C}, \mathcal{X}_s).$$

Of course, we may simply realize the assignment $(\mathcal{C}, (\mathcal{X}/S)) \mapsto \mathrm{H}(\mathcal{C}, (\mathcal{X}/S))$ as a functor

$$\mathrm{H}: \mathbf{Wald}_\infty^{\omega, \mathrm{op}} \times \mathbf{Wald}_{\infty/S}^{\mathrm{cocart}} \rightarrow \mathbf{Kan}$$

by choosing both an equivalence $\mathbf{Wald}_{\infty/S}^{\mathrm{cocart}} \xrightarrow{\sim} \mathrm{Fun}(S, \mathbf{Wald}_\infty)$ and a colimit functor $\mathrm{Fun}(S, \mathbf{Kan}) \rightarrow \mathbf{Kan}$. We have given this explicit construction of the values of this functor in terms of Waldhausen cocartesian fibrations for later use.

In the meantime, we easily have the following.

4.16. Proposition. *If S is a small sifted ∞ -category and if $\mathcal{X} \rightarrow S$ is a Waldhausen cocartesian fibration in which \mathcal{X} is small, then the corresponding functor $\mathrm{H}(-, (\mathcal{X}/S)): \mathbf{Wald}_\infty^{\mathrm{op}} \rightarrow \mathbf{Kan}$ is a virtual Waldhausen ∞ -category.*

4.16.1. Corollary. *If S is a small sifted ∞ -category, then the restriction of H to $\mathbf{Wald}_{\infty/S}^{\mathrm{cocart}}$ factors through the ∞ -category of virtual Waldhausen ∞ -categories:*

$$|\cdot|_S: \mathbf{Wald}_{\infty/S}^{\mathrm{cocart}} \rightarrow \mathbf{VWald}_\infty.$$

4.16.2. Corollary. *Suppose S a small ∞ -category, and suppose $p: \mathcal{X} \rightarrow S$ a Waldhausen cocartesian fibration with \mathcal{X} small. Then $|\mathcal{X}|_S$ is naturally isomorphic in $h\mathbf{VWald}_\infty$ to the colimit of the composite functor*

$$S \xrightarrow{\widetilde{\mathcal{X}}} \mathbf{Wald}_\infty \hookrightarrow \mathbf{VWald}_\infty,$$

wherein $\widetilde{\mathcal{X}}$ is the functor classified by p .

The following is a consequence of [37, Lm. 5.5.8.14].

4.16.3. Corollary. *Suppose \mathcal{X} a virtual Waldhausen ∞ -category. Then there exists a Waldhausen cocartesian fibration $\mathcal{Y} \rightarrow N\Delta^{\mathrm{op}}$ and an equivalence $\mathcal{X} \simeq |\mathcal{Y}|_{N\Delta^{\mathrm{op}}}$.*

4.17. Definition. For any small sifted simplicial set and any Waldhausen cocartesian fibration \mathcal{X}/S , the virtual Waldhausen ∞ -category $|\mathcal{X}|_S$ will be called the *realization* of \mathcal{X}/S .

Part 2. Filtered objects and additive theories

In this part, we study suitable functors from \mathbf{Wald}_∞ to the ∞ -category of pointed objects of an ∞ -topos, which we simply call *theories*. We begin by studying the virtual Waldhausen ∞ -categories of filtered and totally filtered objects of a Waldhausen ∞ -category. Using these, we study the class of *distributive* virtual Waldhausen ∞ -categories; these form a localization of \mathbf{VWald}_∞ , and we show that suspension in this ∞ -category is given by the formation of the virtual Waldhausen ∞ -category of totally filtered objects, which

is in turn an ∞ -categorical analogue of Waldhausen's S_\bullet construction. We then show that suitable excisive functors on the ∞ -category of distributive virtual Waldhausen ∞ -categories correspond to additive theories that satisfy the consequences of an ∞ -categorical analogue of Waldhausen's additivity theorem, and we construct an *additivization* as a Goodwillie derivative, employing our newly minted suspension functor.

5. FILTERED OBJECTS OF WALDHAUSEN ∞ -CATEGORIES

5.1. Notation. Denote by \mathbf{M} the ordinary category whose objects of \mathbf{M} are pairs (\mathbf{m}, i) consisting of an object $\mathbf{m} \in \Delta$ and an element $i \in \mathbf{m}$ and whose morphisms $(\mathbf{n}, j) \rightarrow (\mathbf{m}, i)$ are maps $\phi: \mathbf{m} \rightarrow \mathbf{n}$ of Δ such that $j \leq \phi(i)$. This category comes equipped with a natural projection $\mathbf{M} \rightarrow \Delta^{\text{op}}$.

The nerve $N\mathbf{M}$ can be endowed with a pair structure by setting

$$(NM)_\dagger := NM \times_{N\Delta^{\text{op}}} \iota N\Delta^{\text{op}}.$$

Put differently, an edge of \mathbf{M} is ingressive just in case it covers an equivalence of Δ^{op} . It is easy to see that the projection $\mathbf{M} \rightarrow \Delta^{\text{op}}$ is a Grothendieck fibration, and so the projection $\pi: NM \rightarrow N\Delta^{\text{op}}$ is a pair cartesian fibration; the functor $N\Delta \rightarrow \mathbf{Pair}_\infty$ classified by π assigns to any object $\mathbf{m} \in N\Delta$ the pair $(\Delta^{\mathbf{m}})^\sharp$.

5.2. Construction. Fix an ∞ -category S . For any morphism of pairs $\mathcal{X} \rightarrow S^\flat$, define the simplicial set $\mathcal{F}(\mathcal{X}/S)$ as the simplicial set over $N\Delta^{\text{op}} \times S$ satisfying the following universal property. We require, for any simplicial set K and any map $\sigma: K \rightarrow N\Delta^{\text{op}} \times S$, a bijection

$$\text{Mor}_{/(N\Delta^{\text{op}} \times S)}(K, \mathcal{F}(\mathcal{X}/S)) \cong \text{Mor}_{s\mathbf{Set}(2)/(S, \iota S)}((K \times_{N\Delta^{\text{op}}} NM, K \times_{N\Delta^{\text{op}}} (NM)_\dagger), (\mathcal{X}, \mathcal{X}_\dagger)),$$

functorial in σ . Here, the category $s\mathbf{Set}(2)$ is the one defined in 3.8.

For any object $\mathbf{m} \in \Delta$, write $\mathcal{F}_m(\mathcal{X}/S)$ for the fiber of the projection $\mathcal{F}(\mathcal{X}/S) \rightarrow N\Delta^{\text{op}}$ over \mathbf{m} . When $S = \Delta^0$, write $\mathcal{F}(\mathcal{X})$ for $\mathcal{F}(\mathcal{X}/S)$, and for any object $\mathbf{m} \in \Delta$, write $\mathcal{F}_m(\mathcal{X})$ for the fiber of the projection $\mathcal{F}(\mathcal{X}) \rightarrow N\Delta^{\text{op}}$ over \mathbf{m} .

5.3. Definition. Suppose $\mathcal{X} \rightarrow S^\flat$ a morphism of pairs. Then the vertices of $\mathcal{F}(\mathcal{X}/S)$ will be called *filtered objects of \mathcal{X} over S* ; more specifically, the vertices of $\mathcal{F}_m(\mathcal{X}/S)$ will be called *m -filtered objects of \mathcal{X} over S* . The edges will be called *morphisms of filtered objects of \mathcal{X} over S* .

The following are immediate consequences of 3.9.

5.4. Proposition. *Suppose $\mathcal{X} \rightarrow S$ a pair cocartesian fibration. Then the functor*

$$\mathcal{F}(\mathcal{X}/S) \rightarrow N\Delta^{\text{op}} \times S$$

is a cocartesian fibration.

5.4.1. Corollary. *For any pair cocartesian fibration $\mathcal{X} \rightarrow S$, the simplicial set $\mathcal{F}(\mathcal{X}/S)$ is an ∞ -category.*

5.5. The functor $N\Delta^{\text{op}} \times S \rightarrow \mathbf{Cat}_\infty$ classified by the cocartesian fibration

$$\mathcal{F}(\mathcal{X}/S) \rightarrow N\Delta^{\text{op}} \times S$$

assigns to any object $(\mathbf{m}, s) \in N\Delta^{\text{op}} \times S$ the ∞ -category $\text{Fun}_{\mathbf{Pair}_\infty}^b((\Delta^{\mathbf{m}})^\sharp, \mathcal{X}_s)$ of filtered objects

$$X_0 \rightarrowtail X_1 \rightarrowtail \cdots \rightarrowtail X_m$$

of \mathcal{X}_s and to any morphism $(\phi, f): (\mathbf{m}, s) \rightarrow (\mathbf{n}, t)$ of $N\Delta^{\text{op}} \times S$ the functor

$$\text{Fun}_{\mathbf{Pair}_\infty}^b((\Delta^{\mathbf{m}})^\sharp, \mathcal{X}_s) \rightarrow \text{Fun}_{\mathbf{Pair}_\infty}^b((\Delta^{\mathbf{n}})^\sharp, \mathcal{X}_t)$$

that carries the filtered object above to the filtered object

$$f_\star(X_{\phi(0)}) \rightarrowtail f_\star(X_{\phi(1)}) \rightarrowtail \cdots \rightarrowtail f_\star(X_{\phi(m)})$$

We may endow the ∞ -categories $\mathcal{F}(\mathcal{X}/S)$ of filtered objects with a pair structure in a variety of ways, but we wish to focus on one pair structure that will retain good formal properties when we pass to the subcategory of totally filtered objects.

5.6. Definition. Suppose \mathcal{C} a Waldhausen ∞ -category, and consider the functors

$$s: \mathrm{Fun}_{\mathbf{Pair}_\infty}^b((\Delta^1)^\sharp, \mathcal{C}) \longrightarrow \mathcal{C} \quad \text{and} \quad t: \mathrm{Fun}_{\mathbf{Pair}_\infty}^b((\Delta^1)^\sharp, \mathcal{C}) \longrightarrow \mathcal{C}.$$

Clearly s is a cocartesian fibration. We may now endow the ∞ -category $\mathrm{Fun}_{\mathbf{Pair}_\infty}^b((\Delta^1)^\sharp, \mathcal{C})$ with a pair structure by letting $\mathrm{Fun}_{\mathbf{Pair}_\infty}^b((\Delta^1)^\sharp, \mathcal{C})_\dagger$ be the smallest subcategory of $\mathrm{Fun}_{\mathbf{Pair}_\infty}^b((\Delta^1)^\sharp, \mathcal{C})$ containing the following classes of edges:

- (5.6.1) any edge ϕ such that both $s(\phi)$ is an equivalence and $t(\phi)$ is ingressive and
- (5.6.2) any s -cocartesian edge that covers a cofibration.

Now suppose $p: \mathcal{X} \longrightarrow S$ a Waldhausen cocartesian fibration. We may now endow the ∞ -category $\mathcal{F}(\mathcal{X}/S)$ with a pair structure in the following manner. We let $\mathcal{F}(\mathcal{X}/S)_\dagger \subset \mathcal{F}(\mathcal{X}/S)$ be the smallest subcategory containing all equivalences as well as any edge $\phi: \Delta^1 \longrightarrow \mathcal{X}$ that covers a degenerate edge $\mathrm{id}_{(\mathbf{m}, s)}$ of $N\Delta^{\mathrm{op}} \times S$ — whence it can be identified as a functor $\phi: \Delta^1 \longrightarrow \mathrm{Fun}_{\mathbf{Pair}_\infty}^b((\Delta^m)^\sharp, \mathcal{X}_s)$ — such that for any edge $\eta: \Delta^1 \longrightarrow \Delta^m$, the edge

$$\Delta^1 \xrightarrow{\phi} \mathrm{Fun}_{\mathbf{Pair}_\infty}^b((\Delta^m)^\sharp, \mathcal{X}_s) \xrightarrow{\eta^*} \mathrm{Fun}_{\mathbf{Pair}_\infty}^b((\Delta^1)^\sharp, \mathcal{X}_s)$$

is ingressive in the sense above.

5.7. Proposition. Suppose $p: \mathcal{X} \longrightarrow S$ a Waldhausen cocartesian fibration. Then the functor

$$\mathcal{F}(p): \mathcal{F}(\mathcal{X}/S) \longrightarrow N\Delta^{\mathrm{op}} \times S$$

is a Waldhausen cocartesian fibration.

Proof. With the structure on $\mathcal{F}(\mathcal{X}/S)$ described above, $\mathcal{F}(p)$ is easily seen to be a pair cocartesian fibration.

We claim that for any vertex $(\mathbf{m}, s) \in N\Delta^{\mathrm{op}} \times S$, the pair $\mathrm{Fun}_{\mathbf{Pair}_\infty}^b((\Delta^m)^\sharp, \mathcal{X}_s)$ is a Waldhausen ∞ -category. Note that since \mathcal{X}_s admits a zero object, so does $\mathrm{Fun}_{\mathbf{Pair}_\infty}^b((\Delta^m)^\sharp, \mathcal{X}_s)$. For the remaining two axioms, one reduces immediately to the case where $m = 1$. Then (2.4.2) follows from the presence of (5.6.1) among cofibrations. We prove (2.4.3) in two steps. First, to see that pushouts along cofibrations *exist*, one may note that cofibrations of $\mathrm{Fun}_{\mathbf{Pair}_\infty}^b((\Delta^1)^\sharp, \mathcal{X}_s)$ are in particular cofibrations of $\mathcal{O}(\mathcal{C})$, for which the existence of pushouts is clear. Second, to see that a pushout of a cofibration is again a cofibration, it suffices to see that a pushout of any edge of either of the classes (5.6.1) or (5.6.2) is of the same class. For the class (5.6.1), this follows from the fact that pushouts in $\mathrm{Fun}_{\mathbf{Pair}_\infty}^b((\Delta^1)^\sharp, \mathcal{X}_s)$ are computed pointwise. A pushout of a morphism of the class (5.6.2) is a cube

$$X: (\Delta^1)^\sharp \times (\Delta^1)^\sharp \times (\Delta^1)^\flat \longrightarrow \mathcal{X}_s$$

in which the faces

$$X|((\Delta^1)^\sharp \times (\Delta^1)^\sharp \times \Delta^{\{0\}}), \quad X|((\Delta^1)^\sharp \times \Delta^{\{0\}} \times (\Delta^1)^\flat), \quad \text{and} \quad X|((\Delta^1)^\sharp \times \Delta^{\{0\}} \times (\Delta^1)^\flat)$$

are all pushouts. By Quetzalcoatl, the face $X|((\Delta^1)^\sharp \times (\Delta^1)^\sharp \times \Delta^{\{1\}})$ must be a pushout as well; this is precisely the claim that the pushout is s -cocartesian.

For any $\mathbf{m} \in \Delta$ and any edge $f: s \longrightarrow t$ of S , since the functor $f_{\mathcal{X},!}: \mathcal{X}_s \longrightarrow \mathcal{X}_t$ is exact, it follows directly that the functor $f_{\mathcal{F},!}: \mathrm{Fun}_{\mathbf{Pair}_\infty}^b((\Delta^m)^\sharp, \mathcal{X}_s) \longrightarrow \mathrm{Fun}_{\mathbf{Pair}_\infty}^b((\Delta^m)^\sharp, \mathcal{X}_t)$ is exact as well. Now for any fixed vertex $s \in S_0$ and any simplicial operator $\phi: \mathbf{n} \longrightarrow \mathbf{m}$ of Δ , the functor

$$\phi_{\mathcal{F},!}: \mathrm{Fun}_{\mathbf{Pair}_\infty}^b((\Delta^m)^\sharp, \mathcal{X}_s) \longrightarrow \mathrm{Fun}_{\mathbf{Pair}_\infty}^b((\Delta^n)^\sharp, \mathcal{X}_s)$$

visibly carries cofibrations to cofibrations, and it preserves zero objects as well as any pushouts that exist, since limits and colimits are formed pointwise. \square

Thanks to 3.10, we have:

5.7.1. Corollary. The assignment $(\mathcal{X}/S) \longmapsto \mathcal{F}(\mathcal{X}/S)$ defines a functor

$$\mathcal{F}: \mathbf{Wald}_\infty^{\mathrm{cocart}} \longrightarrow \mathbf{Wald}_\infty^{\mathrm{cocart}}$$

covering the endofunctor $S \longmapsto N\Delta^{\mathrm{op}} \times S$ of \mathbf{Cat}_∞ .

5.8. **Proposition.** Suppose $p: \mathcal{X} \rightarrow S$ a Waldhausen cocartesian fibration, and suppose

$$F_*(\mathcal{X}/S): N\Delta^{\text{op}} \rightarrow \text{Fun}(S, \mathbf{Wald}_\infty)$$

the functor classified by the Waldhausen cocartesian fibration $\mathcal{F}(p): \mathcal{F}(\mathcal{X}/S) \rightarrow N\Delta^{\text{op}} \times S$. Then $F_*(\mathcal{X}/S)$ is a category object [38, Df. 1.1.1]; that is, the maps $\{i-1, i\} \hookrightarrow \mathbf{m}$ of Δ induce morphisms that exhibit $F_m(\mathcal{X}/S)$ as the limit in $\text{Fun}(S, \mathbf{Wald}_\infty)$ of the diagram

$$\begin{array}{ccccccc} F_{\{0,1\}}(\mathcal{X}/S) & & F_{\{1,2\}}(\mathcal{X}/S) & & F_{\{2,3\}}(\mathcal{X}/S) & & F_{\{m-1,m\}}(\mathcal{X}/S) \\ & \searrow & \swarrow & \searrow & \swarrow & \searrow & \swarrow \\ & F_{\{1\}}(\mathcal{X}/S) & & F_{\{2\}}(\mathcal{X}/S) & & \dots & \end{array}$$

Proof. It clearly suffices to assume that $S = \Delta^0$. It is easy to see that $(\Delta^m)^\#$ decomposes in \mathbf{Pair}_∞ as the pushout of the diagram

$$\begin{array}{ccccccc} & & (\Delta^{\{1\}})^\# & & (\Delta^{\{2\}})^\# & & \dots \\ & \swarrow & & \searrow & \swarrow & \searrow & \\ (\Delta^{\{0,1\}})^\# & & & & (\Delta^{\{1,2\}})^\# & & (\Delta^{\{2,3\}})^\# & & \dots & & (\Delta^{\{m-1,m\}})^\# \end{array}$$

since the analogous statement is true in \mathbf{Cat}_∞ . Thus $F_m(\mathcal{X}/S)$ is the desired limit in $\text{Fun}(S, \mathbf{Cat}_\infty)$, and it follows immediately from the definitions of the cofibrations of $F_m(\mathcal{X}/S)$ that $F_m(\mathcal{X}/S)$ is the desired limit in the ∞ -category $\text{Fun}(S, \mathbf{Pair}_\infty)$ and thus also in the ∞ -category $\text{Fun}(S, \mathbf{Wald}_\infty)$. \square

5.9. If $\mathcal{X} \rightarrow S$ is a Waldhausen cocartesian fibration, and if $w\mathcal{X} \subset \mathcal{X}$ is a labeling, then let $w\mathcal{F}(\mathcal{X}/S) \subset \mathcal{F}(\mathcal{X}/S)$ be the subcategory consisting of those maps $X \rightarrow Y$ of $\mathcal{F}_m(\mathcal{X}/S) \cong \text{Fun}^b((\Delta^m)^\#, \mathcal{X}/S)$ (with $\mathbf{m} \in N\Delta^{\text{op}}$) such that for any natural number $i \leq m$, the morphism $X_i \rightarrow Y_i$ is labeled in \mathcal{X} .

Now we wish to isolate a certain class of filtered object.

5.10. **Definition.** Suppose (\mathcal{X}/S) a Waldhausen cocartesian fibration. A filtered object $X: (\Delta^m)^\# \rightarrow \mathcal{X}$ will be said to be *totally filtered* if X_0 is a zero object of some fiber \mathcal{X}_s .

5.11. **Notation.** Suppose (\mathcal{X}/S) a Waldhausen cocartesian fibration. Denote by $\mathcal{S}(\mathcal{X}/S)$ the full subcategory of $\mathcal{F}(\mathcal{X}/S)$ spanned by the totally filtered objects, and for any object \mathbf{m} of Δ , write $\mathcal{S}_m(\mathcal{X}/S)$ for the fiber of $\mathcal{S}(\mathcal{X}/S) \rightarrow N\Delta^{\text{op}}$ over $\mathbf{m} \in \Delta$. When $S = \Delta^0$, write $\mathcal{S}(\mathcal{X})$ for $\mathcal{S}(\mathcal{X}/S)$, and for any object \mathbf{m} of Δ , write $\mathcal{S}_m(\mathcal{X})$ for the fiber of $\mathcal{S}(\mathcal{X}) \rightarrow N\Delta^{\text{op}}$ over $\mathbf{m} \in \Delta$.

The following result is immediate from [37, Pr. 1.2.12.9].

5.12. **Proposition.** Suppose (\mathcal{X}/S) a Waldhausen cocartesian fibration. For any integer $m \geq 0$, the 0-th face map defines an equivalence of ∞ -categories $\mathcal{S}_{1+m}(\mathcal{X}/S) \rightarrow \mathcal{F}_m(\mathcal{X}/S)$, and the map $\mathcal{S}_0(\mathcal{X}/S) \rightarrow S$ is an equivalence.

We may lift the pair structure on $\mathcal{F}_m(\mathcal{X}/S)$ along this equivalence to obtain a pair structure on $\mathcal{S}_{1+m}(\mathcal{X}/S)$. One sees that the inclusion

$$J_m: \mathcal{S}_m(\mathcal{X}/S) \hookrightarrow \mathcal{F}_m(\mathcal{X}/S) \simeq \mathcal{S}_{1+m}(\mathcal{X}/S)$$

is a strict functor of pairs, and we deduce the following.

5.12.1. **Corollary.** Suppose $\mathcal{X} \rightarrow S$ a Waldhausen cocartesian fibration. For any integer $m \geq 0$, the 0-th face map defines an essentially surjective functor $\mathcal{S}_{1+m}(\mathcal{X}/S) \rightarrow \mathcal{S}_m(\mathcal{X}/S)$ that is a left inverse in \mathbf{hPair}_∞ to the inclusion J_m .

We now aim to demonstrate that for any Waldhausen cocartesian fibration $\mathcal{X} \rightarrow S$, the functor $\mathcal{S}(\mathcal{X}/S) \rightarrow N\Delta^{\text{op}} \times S$ is a Waldhausen cocartesian fibration. For this purpose, it is convenient to study the mapping cylinder $\mathcal{M}(\mathcal{X}/S)$ of the functor $\mathcal{S}(\mathcal{X}/S) \rightarrow \mathcal{F}(\mathcal{X}/S)$.

5.13. Notation. For any Waldhausen cocartesian fibration $\mathcal{X} \rightarrow S$, write $\mathcal{M}(\mathcal{X}/S)$ for the full subcategory of $\Delta^1 \times \mathcal{F}(\mathcal{X}/S)$ spanned by those pairs (i, X) such that X is totally filtered if $i = 1$. This ∞ -category comes equipped with an inner fibration

$$\mathcal{M}(\mathcal{X}/S) \rightarrow \Delta^1 \times N\Delta^{\text{op}} \times S.$$

Let $\mathcal{M}(\mathcal{X}/S)_{\dagger} \subset \mathcal{M}(\mathcal{X}/S)$ be the subcategory whose edges are maps $(i, X) \rightarrow (j, Y)$ such that $i = j$ and $X \rightarrow Y$ is a cofibration of $\mathcal{F}(\mathcal{X}/S)$.

Our first lemma is obvious by construction.

5.14. Lemma. *For any Waldhausen cocartesian fibration $\mathcal{X} \rightarrow S$, the natural projection $\mathcal{M}(\mathcal{X}/S) \rightarrow \Delta^1$ is a pair cartesian fibration.*

Our next lemma, however, is subtler.

5.15. Lemma. *For any Waldhausen cocartesian fibration $\mathcal{X} \rightarrow S$, the natural projection $\mathcal{M}(\mathcal{X}/S) \rightarrow \Delta^1$ is a pair cocartesian fibration.*

Proof. By [37, 2.4.1.3(3)], it suffices to show that for any vertex $(\mathbf{m}, s) \in (N\Delta^{\text{op}} \times S)_0$, the inner fibration

$$q: \mathcal{M}_m(\mathcal{X}_s) \rightarrow \Delta^1$$

is a pair cocartesian fibration. Note that an edge $X \rightarrow Y$ of $\mathcal{M}_m(\mathcal{X}_s)$ covering the nondegenerate edge σ of Δ^1 is q -cocartesian if and only if it is an initial object of the fiber $\mathcal{M}_m(\mathcal{X}_s)_{X/} \times_{\Delta_{0/}^1} \{\sigma\}$. If $m = 0$, then the map

$$\mathcal{M}_0(\mathcal{X}_s)_{X/} \rightarrow \Delta_{0/}^1$$

is a trivial fibration [37, Pr. 1.2.12.9], so the fiber over σ is a contractible Kan complex. Let us now induct on m ; assume that $m > 0$ and that the functor $p: \mathcal{M}_{m-1}(\mathcal{X}_s) \rightarrow \Delta^1$ is a cocartesian fibration. It is easy to see that the inclusion $\{0, 1, \dots, m-1\} \hookrightarrow \mathbf{m}$ induces an inner fibration $\phi: \mathcal{M}_m(\mathcal{X}_s) \rightarrow \mathcal{M}_{m-1}(\mathcal{X}_s)$ such that $q = p \circ \phi$. Again by [37, 2.4.1.3(3)], it suffices to show that for any object X of $\mathcal{M}_m(\mathcal{X}_s)$ and any p -cocartesian edge $\eta: \phi(X) \rightarrow Y'$ covering σ , there exists a ϕ -cocartesian edge $X \rightarrow Y$ of $\mathcal{M}_m(\mathcal{X}_s)$ covering η . But this follows directly from (2.4.3).

We now show that q is a *pair* cocartesian fibration. Suppose

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

is a square of $\mathcal{M}_m(\mathcal{X}_s)$ in which $X' \rightarrow X$ and $Y' \rightarrow Y$ are q -cocartesian morphisms and $X \rightarrow Y$ is a cofibration. We aim to show that for any edge $\eta: \Delta^{\{p, q\}} \rightarrow \Delta^m$, the morphism $X'|_{\Delta^{\{p, q\}}} \rightarrow Y'|_{\Delta^{\{p, q\}}}$ is a cofibration. For this, we may factor $X \rightarrow Y$ as

$$X \rightrightarrows Z \rightrightarrows Y,$$

where $Z|_{\Delta^{\{0, \dots, p\}}} = Y|_{\Delta^{\{0, \dots, p\}}}$, and for any $r > p$, the edge $X|_{\Delta^{\{p, r\}}} \rightarrow Z|_{\Delta^{\{p, r\}}}$ is cocartesian. Now choose a cocartesian morphism $Z' \rightarrow Z$ as well. The proof is now completed by the following observations.

(5.15.1) Since the morphism $X|_{\Delta^{\{p, q\}}} \rightarrow Z|_{\Delta^{\{p, q\}}}$ is of type (5.6.2), it follows by Quetzalcoatl that the morphism $X'|_{\Delta^{\{p, q\}}} \rightarrow Z'|_{\Delta^{\{p, q\}}}$ is of type (5.6.2) as well.

(5.15.2) The morphism $Z|_{\Delta^{\{p, q\}}} \rightarrow Y|_{\Delta^{\{p, q\}}}$ is of type (5.6.1) and the morphism $Z'_p \rightarrow X'_p$ is an equivalence; so again by Quetzalcoatl, the morphism $Z'|_{\Delta^{\{p, q\}}} \rightarrow Y'|_{\Delta^{\{p, q\}}}$ is of type (5.6.1). \square

Together, these lemmas exhibit, for any Waldhausen cocartesian fibration $\mathcal{X} \rightarrow S$, an adjunction

$$F: \mathcal{F}(\mathcal{X}/S) \rightleftarrows \mathcal{S}(\mathcal{X}/S): J$$

over $N\Delta^{\text{op}} \times S$ in which both F and J are functors of pairs.

5.16. **Theorem.** Suppose $\mathcal{X} \rightarrow S$ a Waldhausen cocartesian fibration. Then the functor

$$\mathcal{S}(\mathcal{X}/S) \rightarrow N\Delta^{\text{op}} \times S$$

is a Waldhausen cocartesian fibration.

Proof. We first show that the functor $\mathcal{S}(\mathcal{X}/S) \rightarrow N\Delta^{\text{op}} \times S$ is a cocartesian fibration by proving the stronger assertion that the inner fibration

$$p: \mathcal{M}(\mathcal{X}/S) \rightarrow \Delta^1 \times N\Delta^{\text{op}} \times S$$

is a cocartesian fibration. By 5.7, the map

$$(5.16.1) \quad \Delta^{\{0\}} \times_{\Delta^1} \mathcal{M}(\mathcal{X}/S) \rightarrow \Delta^{\{0\}} \times N\Delta^{\text{op}} \times S$$

is a cocartesian fibration. By 5.15, for any vertex $(\mathbf{m}, s) \in (N\Delta^{\text{op}} \times S)_0$, the map

$$(5.16.2) \quad \mathcal{M}(\mathcal{X}/S) \times_{N\Delta^{\text{op}} \times S} \{(\mathbf{m}, s)\} \rightarrow \Delta^1 \times \{(\mathbf{m}, s)\}$$

is a cocartesian fibration. Finally, for any $\mathbf{m} \in \Delta$ and any edge $f: s \rightarrow t$ of S , the functor $f_!: \mathcal{X}_s \rightarrow \mathcal{X}_t$ carries zero objects to zero objects; consequently, any cocartesian edge of $\mathcal{F}(\mathcal{X}/S)$ that covers $(\text{id}_{\mathbf{m}}, f)$ lies in $\mathcal{S}(\mathcal{X}/S)$ if and only if its source does. Thus the map

$$(\Delta^{\{1\}} \times \{\mathbf{m}\}) \times_{\Delta^1 \times N\Delta^{\text{op}}} \mathcal{M}(\mathcal{X}/S) \rightarrow \Delta^{\{1\}} \times \{\mathbf{m}\} \times S$$

is a cocartesian fibration.

Now to complete the proof that p is a cocartesian fibration, thanks to [37, 2.4.1.3(3)] it remains to show that for any vertex $s \in S_0$, any simplicial operator $\phi: \mathbf{n} \rightarrow \mathbf{m}$, and any totally m -filtered object X of \mathcal{X}_s , then there exists a p -cartesian morphism $(1, X) \rightarrow (1, Y)$ of $\mathcal{F}(\mathcal{X}/S)$ covering $(\text{id}_1, \phi, \text{id}_s)$. Write σ for the non-degenerate edge of Δ^1 . The p -cartesian edge $e: (0, X) \rightarrow (1, X)$ covering $(\sigma, \text{id}_{\mathbf{m}}, \text{id}_s)$ is also p -cocartesian. Since (5.16.1) is a cocartesian fibration, there exists a p -cocartesian edge $\eta': (0, X) \rightarrow (0, Y')$ covering $(\text{id}_0, \phi, \text{id}_s)$. Since (5.16.2) is a cocartesian fibration, there exists a p -cocartesian edge $e': (0, Y') \rightarrow (1, Y)$ covering $(\sigma, \text{id}_{\mathbf{n}}, \text{id}_s)$. Since e is p -cocartesian, we have a diagram $\Delta^1 \times \Delta^1 \rightarrow \mathcal{M}(\mathcal{X}/S) \times_S \{s\}$ of the form

$$\begin{array}{ccc} (0, X) & \xrightarrow{\eta'} & (0, Y') \\ e \downarrow & & \downarrow e' \\ (1, X) & \xrightarrow{\eta} & (1, Y). \end{array}$$

It follows from [37, 2.4.1.7] that η is p -cocartesian.

From 5.12 and 5.7 it follows that the fibers of $\mathcal{S}(\mathcal{X}/S) \rightarrow N\Delta^{\text{op}} \times S$ are all Waldhausen ∞ -categories. For any $\mathbf{m} \in \Delta$ and any edge $f: s \rightarrow t$ of S , the functor $f_{\mathcal{X},!}: \mathcal{X}_s \rightarrow \mathcal{X}_t$ is exact, whence it follows by 5.12 that the functor

$$f_{\mathcal{S},!}: \mathcal{S}_m(\mathcal{X}_s) \simeq \text{Fun}_{\mathbf{Pair}_{\infty}}^b((\Delta^{m-1})^{\sharp}, \mathcal{X}_s) \rightarrow \text{Fun}_{\mathbf{Pair}_{\infty}}^b((\Delta^{m-1})^{\sharp}, \mathcal{X}_t) \simeq \mathcal{S}_m(\mathcal{X}_t)$$

is exact, just as in the proof of 5.7. Now for any fixed vertex $s \in S_0$ and any simplicial operator $\phi: \mathbf{n} \rightarrow \mathbf{m}$ of Δ , the functor $\phi_{\mathcal{S},!}: \mathcal{S}_m(\mathcal{X}_s) \rightarrow \mathcal{S}_n(\mathcal{X}_s)$ is by construction the composite

$$\mathcal{S}_m(\mathcal{X}_s) \xrightarrow{J_{m,s}} \mathcal{F}_m(\mathcal{X}_s) \xrightarrow{\phi_{\mathcal{F},!}} \mathcal{F}_n(\mathcal{X}_s) \xrightarrow{F_{n,s}} \mathcal{S}_n(\mathcal{X}_s),$$

and as $\phi_{\mathcal{F},!}$ is an exact functor (5.7), we are reduced to checking that the functors of pairs $J_{m,s}$ and $F_{n,s}$ are each exact functors.

For this, it is clear that $J_{m,s}$ and $F_{n,s}$ each carry zero objects to zero objects, and as $F_{n,s}$ is a left adjoint, it preserves any pushout squares that exist in $\mathcal{F}_n(\mathcal{X}_s)$. Moreover, a pushout square in $\mathcal{S}_m(\mathcal{X}_s)$ is nothing more than a pushout square in $\mathcal{F}_m(\mathcal{X}_s)$ of totally m -filtered objects; hence $J_{m,s}$ preserves pushouts along cofibrations. \square

For any Waldhausen cocartesian fibration $\mathcal{X} \rightarrow S$, write

$$S_*(\mathcal{X}): N\Delta^{\text{op}} \times S \rightarrow \mathbf{Wald}_{\infty}$$

for a functor that classifies the Waldhausen cocartesian fibration $\mathcal{S}(\mathcal{X}/S) \rightarrow N\Delta^{\text{op}} \times S$, and, similarly, write

$$F_*(\mathcal{X}): N\Delta^{\text{op}} \times S \rightarrow \mathbf{Wald}_\infty$$

for a functor that classifies the Waldhausen cocartesian fibration $\mathcal{F}(\mathcal{X}/S) \rightarrow N\Delta^{\text{op}} \times S$. An instant consequence of the construction of the functoriality of \mathcal{S} in the proof above is the following.

5.16.1. **Corollary.** *The functors $F_m: \mathcal{F}_m(\mathcal{X}/S) \rightarrow \mathcal{S}_m(\mathcal{X}/S)$ assemble to a natural transformation*

$$F: F_*(\mathcal{X}) \rightarrow S_*(\mathcal{X}).$$

Note, however, that it is *not* the case that the functors J_m assemble to a natural transformation of this kind.

5.17. For any Waldhausen cocartesian fibration $\mathcal{X} \rightarrow S$, the functor $N\Delta^{\text{op}} \times S \rightarrow \mathbf{Wald}_\infty$ classified by the Waldhausen cocartesian fibration $\mathcal{S}(\mathcal{X}/S) \rightarrow N\Delta^{\text{op}} \times S$ assigns to any object (\mathbf{m}, s) the ∞ -category of totally filtered objects

$$0 \simeq X_0 \twoheadrightarrow X_1 \twoheadrightarrow \cdots \twoheadrightarrow X_m$$

of \mathcal{X}_s . For any morphism $(\phi, f): (\mathbf{m}, s) \rightarrow (\mathbf{n}, t)$ of $N\Delta^{\text{op}} \times S$, the induced functor carries the totally filtered object X above to a representative of the totally filtered object

$$0 \simeq f_*(X_{\phi(0)}/X_{\phi(0)}) \twoheadrightarrow f_*(X_{\phi(1)}/X_{\phi(0)}) \twoheadrightarrow \cdots \twoheadrightarrow f_*(X_{\phi(n)}/X_{\phi(0)}).$$

Giving a definition directly in this style requires solving certain homotopy-coherence problems. Waldhausen showed that the solving these coherence problems amounts to making compatible choices of successive quotients. Lurie also makes use of Waldhausen's idea in [41, §1.2.2]. We have chosen instead to avoid these issues by means of the theory of fibrations.

Thanks to 3.10, the assignments

$$(\mathcal{X}/S) \mapsto (\mathcal{F}(\mathcal{X}/S)/(N\Delta^{\text{op}} \times S)) \quad \text{and} \quad (\mathcal{X}/S) \mapsto (\mathcal{S}(\mathcal{X}/S)/(N\Delta^{\text{op}} \times S))$$

define endofunctors of $\mathbf{Wald}_\infty^{\text{cocart}}$ over the endofunctor $S \mapsto N\Delta^{\text{op}} \times S$ of \mathbf{Cat}_∞ . We now aim to descend these functors to endofunctors of the ∞ -category of virtual Waldhausen ∞ -categories.

5.18. **Lemma.** *The functors $\mathbf{Wald}_\infty \rightarrow \mathbf{Wald}_{\infty/N\Delta^{\text{op}}}^{\text{cocart}}$ given by*

$$\mathcal{C} \mapsto (\mathcal{F}(\mathcal{C})/N\Delta^{\text{op}}) \quad \text{and} \quad \mathcal{C} \mapsto (\mathcal{S}(\mathcal{C})/N\Delta^{\text{op}})$$

are each ω -continuous.

Proof. By Cor. 3.16.1, it is enough to check the claim fiberwise. The assignment $\mathcal{C} \mapsto \mathcal{S}_0(\mathcal{C})$ is an essentially constant functor whose values are all terminal objects; hence since filtered simplicial sets are weakly contractible, this functor preserves filtered colimits. We are now reduced to the claim that for any natural number m , the assignment $\mathcal{C} \mapsto \mathcal{F}_m(\mathcal{C})$ defines a functor $\mathbf{Wald}_\infty \rightarrow \mathbf{Wald}_\infty$ that preserves filtered colimits.

Suppose now that Λ is a filtered simplicial set; by [37, Pr. 5.3.1.16], we may assume that Λ is the nerve of a filtered poset. Suppose $\mathcal{C}: \Lambda^\triangleright \rightarrow \mathbf{Wald}_\infty$ a colimit digram of Waldhausen ∞ -categories. Suppose $\widetilde{\mathcal{F}}_m(\mathcal{C}): \Lambda^\triangleright \rightarrow \mathbf{Pair}_\infty$ be a colimit diagram such that $\widetilde{\mathcal{F}}_m(\mathcal{C})|_\Lambda = \mathcal{F}_m(\mathcal{C})|_\Lambda$. By 4.4, we are reduced to showing that the natural functor of pairs

$$\nu: \widetilde{\mathcal{F}}_m(\mathcal{C})_\infty \rightarrow \mathcal{F}_m(\mathcal{C}_\infty)$$

is an equivalence. Indeed, ν induces an equivalence of the underlying ∞ -categories, since $(\Delta^m)^\sharp \times (\Delta^n)^\flat$ is a compact object of \mathbf{Pair}_∞ ; hence it remains to show that ν is a strict functor of pairs. For this it suffices to show that for any cofibration $\psi: X \twoheadrightarrow Y$ of $\mathcal{F}(\mathcal{C}_\infty)$, there exists a vertex $\alpha \in \Lambda$ and an edge $\bar{\psi}: \bar{X} \rightarrow \bar{Y}$ of $\mathcal{F}(\mathcal{C}_\alpha)$ lifting ψ . It is enough to assume that $m = 1$ and to show that ψ is either of type (5.6.1) or of type (5.6.2). That is, we may assume that ψ is represented by a square

$$(5.18.1) \quad \begin{array}{ccc} X & \twoheadrightarrow & Y \\ \downarrow & & \downarrow \\ X' & \twoheadrightarrow & Y' \end{array}$$

of cofibrations such that either $X \twoheadrightarrow X'$ is an equivalence or else the square (5.18.1) is a pushout. Since $\Delta^1 \times \Delta^1$ is a compact ∞ -category, a square of cofibrations of the form (5.18.1) must lift to a square of cofibrations

$$(5.18.2) \quad \begin{array}{ccc} \overline{X} & \twoheadrightarrow & \overline{Y} \\ \downarrow & & \downarrow \\ \overline{X}' & \twoheadrightarrow & \overline{Y}' \end{array}$$

of \mathcal{C}_α for some vertex $\alpha \in \Lambda$. Now the argument is completed by the following brace of observations.

(5.18.3) If $X \twoheadrightarrow X'$ is an equivalence, then, increasing α if necessary, we may assume that its lift $\overline{X} \twoheadrightarrow \overline{X}'$ in \mathcal{C}_α is an equivalence as well, since for example the pushout

$$\Delta^3 \cup (\Delta^{\{0,2\}} \sqcup \Delta^{\{1,3\}}) (\Delta^0 \sqcup \Delta^0)$$

is compact in the Joyal model structure; hence it represents a cofibration of type (5.6.1) of $\mathcal{F}_1(\mathcal{C}_\alpha)$.

(5.18.4) If (5.18.1) is a pushout, then one may form the pushout of $\overline{X}' \leftarrow \overline{X} \twoheadrightarrow \overline{Y}$ in \mathcal{C}_α . Since $\mathcal{C}_\alpha \rightarrow \mathcal{C}_\infty$ preserves such pushouts, we may assume that (5.18.2) is a pushout square in \mathcal{C}_α ; hence it represents a cofibration of type (5.6.2) of $\mathcal{F}_1(\mathcal{C}_\alpha)$. \square

5.19. Construction. One may compose the functors

$$\mathcal{F}: \mathbf{Wald}_\infty \longrightarrow \mathbf{Wald}_{\infty, /N\Delta^{\text{op}}} \quad \text{and} \quad \mathcal{S}: \mathbf{Wald}_\infty \longrightarrow \mathbf{Wald}_{\infty, /N\Delta^{\text{op}}}$$

with the functor $|\cdot|_{N\Delta^{\text{op}}}$; the results are models for the functors $\mathbf{Wald}_\infty \rightarrow \mathbf{VWald}_\infty$ that assign to any Waldhausen ∞ -category the geometric realizations of the simplicial virtual Waldhausen ∞ -categories $\mathcal{F}_*(\mathcal{C})$ and $\mathcal{S}_*(\mathcal{C})$. In particular, these composites are ω -continuous functors $\mathbf{Wald}_\infty \rightarrow \mathbf{VWald}_\infty$, whence one obtains essentially unique endofunctors \mathcal{F} and \mathcal{S} of \mathbf{VWald}_∞ that preserve sifted colimits such that the squares

$$\begin{array}{ccc} \mathbf{Wald}_\infty & \xrightarrow{\mathcal{F}} & \mathbf{Wald}_{\infty, /N\Delta^{\text{op}}}^{\text{cocart}} \\ \downarrow j & & \downarrow |\cdot|_{N\Delta^{\text{op}}} \\ \mathbf{VWald}_\infty & \xrightarrow{\mathcal{F}} & \mathbf{VWald}_\infty \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbf{Wald}_\infty & \xrightarrow{\mathcal{S}} & \mathbf{Wald}_{\infty, /N\Delta^{\text{op}}}^{\text{cocart}} \\ \downarrow j & & \downarrow |\cdot|_{N\Delta^{\text{op}}} \\ \mathbf{VWald}_\infty & \xrightarrow{\mathcal{S}} & \mathbf{VWald}_\infty \end{array}$$

commute via a specified homotopy.

5.20. Now the natural transformation F from Cor. 5.16.1 descends further to a natural transformation $F: \mathcal{F} \rightarrow \mathcal{S}$ of endofunctors of \mathbf{VWald}_∞ .

As it happens, the functor $\mathcal{F}: \mathbf{VWald}_\infty \rightarrow \mathbf{VWald}_\infty$ is not particularly exciting:

5.21. Proposition. *For any virtual Waldhausen ∞ -category \mathcal{X} , the virtual Waldhausen ∞ -category $\mathcal{F}(\mathcal{X})$ is terminal.*

Proof. For any Waldhausen ∞ -category \mathcal{C} , the virtual Waldhausen ∞ -category $|\mathcal{F}(\mathcal{C})|_{N\Delta^{\text{op}}}$ is by definition a functor $\mathbf{Wald}_\infty^\omega \rightarrow \mathbf{Kan}$ that assigns to any compact Waldhausen ∞ -category \mathcal{Y} the geometric realization of the simplicial space

$$\mathbf{m} \mapsto \mathbf{Wald}_\infty^\Delta(\mathcal{Y}, \mathcal{F}_m(\mathcal{C})).$$

By 5.12, this is the path space of the simplicial space $\mathbf{m} \mapsto \mathbf{Wald}_\infty^\Delta(\mathcal{Y}, \mathcal{S}_m(\mathcal{C}))$. \square

This result permits us to regard the virtual Waldhausen ∞ -category $\mathcal{F}(\mathcal{X})$ as a *cone* on the virtual Waldhausen ∞ -category \mathcal{X} . With this perspective, we will view the induced morphism $F: \mathcal{F}(\mathcal{X}) \rightarrow \mathcal{S}(\mathcal{X})$ induced by the functor F as a suitable “quotient” of $\mathcal{F}(\mathcal{X})$ that identifies $\mathcal{S}(\mathcal{X})$ as a *suspension* of \mathcal{X} in a suitable ∞ -category. We shall return to this point in the next section.

The essential unicity of the extensions \mathcal{F} and \mathcal{S} to \mathbf{VWald}_∞ now implies the following.

5.22. **Proposition.** *If S is a small sifted ∞ -category, then the squares*

$$\begin{array}{ccc} \mathbf{Wald}_{\infty/S}^{\text{cocart}} & \xrightarrow{\mathcal{F}} & \mathbf{Wald}_{\infty/N\Delta^{\text{op}} \times S}^{\text{cocart}} \\ \downarrow |\cdot|_S & & \downarrow |\cdot|_{N\Delta^{\text{op}} \times S} \\ \mathbf{VWald}_{\infty} & \xrightarrow{\mathcal{F}} & \mathbf{VWald}_{\infty} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbf{Wald}_{\infty/S}^{\text{cocart}} & \xrightarrow{\mathcal{S}} & \mathbf{Wald}_{\infty/N\Delta^{\text{op}} \times S}^{\text{cocart}} \\ \downarrow |\cdot|_S & & \downarrow |\cdot|_{N\Delta^{\text{op}} \times S} \\ \mathbf{VWald}_{\infty} & \xrightarrow{\mathcal{S}} & \mathbf{VWald}_{\infty} \end{array}$$

commute via a specified homotopy.

Of course this is no surprise for $\mathcal{F}: \mathbf{VWald}_{\infty} \rightarrow \mathbf{VWald}_{\infty}$, as we have already seen that \mathcal{F} is an essentially constant functor whose value at any object is terminal.

6. DISTRIBUTIVE VIRTUAL WALDHAUSEN ∞ -CATEGORIES

If \mathcal{E} is an ∞ -category that admits all small sifted colimits, then a functor $f: \mathbf{Wald}_{\infty} \rightarrow \mathcal{E}$ may be described and studied through its left derived functor (4.14)

$$F: \mathbf{VWald}_{\infty} \rightarrow \mathcal{E}.$$

We now construct a suitable subcategory of the ∞ -category \mathbf{VWald}_{∞} in order to characterize Waldhausen's notion of *additivity* in a number of equivalent ways.

6.1. **Construction.** Suppose \mathcal{C} a Waldhausen category. For any integer $m \geq 0$, one has, corresponding to the unique edge $\mathbf{m} \rightarrow \mathbf{0}$ of Δ , exact functors

$$E_m: \mathcal{C} \simeq \mathcal{F}_0(\mathcal{C}) \rightarrow \mathcal{F}_m(\mathcal{C}) \quad \text{and} \quad E'_m: 0 \simeq \mathcal{S}_0(\mathcal{C}) \rightarrow \mathcal{S}_m(\mathcal{C}).$$

We have a commutative square

$$\begin{array}{ccc} \mathcal{F}_0(\mathcal{C}) & \xrightarrow{F_0} & \mathcal{S}_0(\mathcal{C}) \\ E_m \downarrow & & \downarrow E'_m \\ \mathcal{F}_m(\mathcal{C}) & \xrightarrow{F_m} & \mathcal{S}_m(\mathcal{C}) \end{array}$$

The functor E_m may equivalently be described as the m -th component of the counit of an adjunction

$$C: \mathbf{Wald}_{\infty} \rightleftarrows \mathbf{Wald}_{\infty/N\Delta^{\text{op}}}^{\text{cocart}}: R,$$

where C is the “constant” functor $\mathcal{C} \mapsto \mathcal{C} \times N\Delta^{\text{op}}$; hence E_m is natural in \mathcal{C} and functorial in $\mathbf{m} \in N\Delta^{\text{op}}$. Consequently there is an induced natural transformation $E: \text{id} \rightarrow \mathcal{F}$ of endofunctors of \mathbf{VWald}_{∞} , and the square above descends to a square of natural transformations

$$(6.1.1) \quad \begin{array}{ccc} \text{id} & \longrightarrow & 0 \\ E \downarrow & & \downarrow \\ \mathcal{F} & \xrightarrow{F} & \mathcal{S}, \end{array}$$

where 0 is the essentially constant endofunctor of \mathbf{VWald}_{∞} whose value at any object is a zero object.

6.2. **Definition.** A presheaf $\mathcal{X} \in \mathcal{P}(\mathbf{Wald}_{\infty}^{\omega})$ is *distributive* if for every compact Waldhausen ∞ -category \mathcal{C} and every integer $m \geq 0$, the exact functors E_m and J_m induce functors

$$\mathcal{X}(\mathcal{F}_m(\mathcal{C})) \rightarrow \mathcal{X}(\mathcal{C}) \quad \text{and} \quad \mathcal{X}(\mathcal{F}_m(\mathcal{C})) \rightarrow \mathcal{X}(\mathcal{S}_m(\mathcal{C}))$$

that together exhibit $\mathcal{X}(\mathcal{F}_m(\mathcal{C}))$ as the product of $\mathcal{X}(\mathcal{C})$ and $\mathcal{X}(\mathcal{S}_m(\mathcal{C}))$.

6.3. **Lemma.** *A presheaf $\mathcal{X} \in \mathcal{P}(\mathbf{Wald}_{\infty}^{\omega})$ is distributive only if \mathcal{X} carries direct sums in $\mathbf{Wald}_{\infty}^{\omega}$ to products — that is, only if \mathcal{X} is a virtual Waldhausen ∞ -category.*

Proof. Suppose \mathcal{C} and \mathcal{D} two compact Waldhausen ∞ -categories. Consider the retract diagrams

$$\begin{array}{ccccc} \mathcal{C} & \longrightarrow & \mathcal{C} \oplus \mathcal{D} & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow E_1 & & \downarrow \\ \mathcal{C} \oplus \mathcal{D} & \xrightarrow{E_1 \oplus J_1} & \mathcal{F}_1(\mathcal{C} \oplus \mathcal{D}) & \xrightarrow{I_{1,0} \oplus F_1} & \mathcal{C} \oplus \mathcal{D} \end{array}$$

and

$$\begin{array}{ccccc} \mathcal{D} & \longrightarrow & \mathcal{C} \oplus \mathcal{D} & \longrightarrow & \mathcal{D} \\ \downarrow & & \downarrow J_1 & & \downarrow \\ \mathcal{C} \oplus \mathcal{D} & \xrightarrow{E_1 \oplus J_1} & \mathcal{F}_1(\mathcal{C} \oplus \mathcal{D}) & \xrightarrow{I_{1,0} \oplus F_1} & \mathcal{C} \oplus \mathcal{D}. \end{array}$$

Here $I_{1,0}$ is the functor induced by the morphism $0 \mapsto 0$. For any distributive virtual Waldhausen ∞ -category \mathcal{X} , we have an induced retract diagram

$$(6.3.1) \quad \begin{array}{ccccc} \mathcal{X}(\mathcal{C} \oplus \mathcal{D}) & \longrightarrow & \mathcal{X}(\mathcal{F}_1(\mathcal{C} \oplus \mathcal{D})) & \longrightarrow & \mathcal{X}(\mathcal{C} \oplus \mathcal{D}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{X}(\mathcal{C}) \times \mathcal{X}(\mathcal{D}) & \longrightarrow & \mathcal{X}(\mathcal{C} \oplus \mathcal{D}) \times \mathcal{X}(\mathcal{C} \oplus \mathcal{D}) & \longrightarrow & \mathcal{X}(\mathcal{C}) \times \mathcal{X}(\mathcal{D}). \end{array}$$

Since the center vertical map is an equivalence, and since equivalences are closed under retracts, so are the outer vertical maps. \square

6.4. Notation. Denote by $\mathbf{V}_{\text{add}}\mathbf{Wald}_{\infty}$ the full subcategory of \mathbf{VWald}_{∞} spanned by the distributive functors.

Since sifted colimits in \mathbf{VWald}_{∞} commute with products [37, Lm. 5.5.8.11], we deduce the following.

6.5. Lemma. *The subcategory $\mathbf{V}_{\text{add}}\mathbf{Wald}_{\infty} \subset \mathbf{VWald}_{\infty}$ is stable under sifted colimits.*

6.6. Note that representable presheaves are typically *not* distributive. Consequently, the obvious fully faithful inclusion $\mathbf{Wald}_{\infty}^{\omega} \hookrightarrow \mathbf{VWald}_{\infty}$ does not factor through $\mathbf{V}_{\text{add}}\mathbf{Wald}_{\infty} \subset \mathbf{VWald}_{\infty}$.

6.7. Proposition. *The inclusion functor admits a left adjoint*

$$L^{\text{add}}: \mathbf{VWald}_{\infty} \longrightarrow \mathbf{V}_{\text{add}}\mathbf{Wald}_{\infty},$$

which exhibits $\mathbf{V}_{\text{add}}\mathbf{Wald}_{\infty}$ as an accessible localization of \mathbf{VWald}_{∞} .

Proof. For any compact Waldhausen ∞ -category \mathcal{C} and every integer $m \geq 0$, consider the exact functor

$$E_m \oplus J_m: \mathcal{C} \oplus \mathcal{S}_m(\mathcal{C}) \longrightarrow \mathcal{F}_m(\mathcal{C});$$

let S be the set of morphisms of \mathbf{VWald}_{∞} of this form; let \bar{S} be the strongly saturated class it generates. Since $\mathbf{Wald}_{\infty}^{\omega}$ is essentially small, the class \bar{S} is of small generation. Hence we may form the accessible localization $S^{-1}\mathbf{VWald}_{\infty}$. Since virtual Waldhausen ∞ -categories are functors $\mathcal{X}: \mathbf{Wald}_{\infty}^{\omega, \text{op}} \rightarrow \mathbf{Kan}$ that preserve products, one sees that $S^{-1}\mathbf{VWald}_{\infty}$ coincides with the full subcategory $\mathbf{V}_{\text{add}}\mathbf{Wald}_{\infty} \subset \mathbf{VWald}_{\infty}$. \square

The fully faithful inclusion $\mathbf{V}_{\text{add}}\mathbf{Wald}_{\infty} \hookrightarrow \mathbf{VWald}_{\infty}$ preserve finite products, and its left adjoint L^{add} preserve finite coproducts, whence we deduce the following.

6.7.1. Corollary. *The ∞ -category $\mathbf{V}_{\text{add}}\mathbf{Wald}_{\infty}$ is compactly generated and admits finite direct sums, which are preserved by the inclusion $\mathbf{V}_{\text{add}}\mathbf{Wald}_{\infty} \hookrightarrow \mathbf{VWald}_{\infty}$*

Combining this with Lm. 6.5 and [41, Lm. 1.3.2.9], we deduce the following somewhat surprising fact.

6.7.2. Corollary. *The subcategory $\mathbf{V}_{\text{add}}\mathbf{Wald}_{\infty} \subset \mathbf{VWald}_{\infty}$ is stable under all small colimits.*

A large portion of the usefulness of the ∞ -category $\mathbf{V}_{\text{add}}\mathbf{Wald}_{\infty}$ of distributive virtual Waldhausen ∞ -categories is derived from the relationship it bears to the endofunctor \mathcal{S} on \mathbf{VWald}_{∞} . The first indication of such a close relationship is the following result.

6.8. **Proposition.** *The diagram*

$$\begin{array}{ccc} \mathbf{VWald}_\infty & \xrightarrow{\mathcal{S}} & \mathbf{VWald}_\infty \\ L^{\text{add}} \downarrow & & \downarrow L^{\text{add}} \\ \mathbf{V}_{\text{add}} \mathbf{Wald}_\infty & \xrightarrow{\Sigma_{\mathbf{V}_{\text{add}} \mathbf{Wald}_\infty}} & \mathbf{V}_{\text{add}} \mathbf{Wald}_\infty \end{array}$$

commutes (up to homotopy), where $\Sigma_{\mathbf{V}_{\text{add}} \mathbf{Wald}_\infty}$ is the suspension endofunctor on $\mathbf{V}_{\text{add}} \mathbf{Wald}_\infty$.

Proof. Apply L^{add} to the square (6.1.1) to obtain a square

$$(6.8.1) \quad \begin{array}{ccc} L^{\text{add}} & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ L^{\text{add}} \circ \mathcal{F} & \xrightarrow{F} & L^{\text{add}} \circ \mathcal{S}, \end{array}$$

of natural transformations between functors $\mathbf{VWald}_\infty \rightarrow \mathbf{V}_{\text{add}} \mathbf{Wald}_\infty$. Since \mathcal{F} is essentially constant with value the zero object, this gives rise to a natural transformation $\Sigma \circ L^{\text{add}} \rightarrow L^{\text{add}} \circ \mathcal{S}$. To see that this natural transformation is an equivalence, it suffices to consider its value on a compact Waldhausen ∞ -category \mathcal{C} . Now for any $\mathbf{m} \in N\Delta^{\text{op}}$, we have a diagram

$$\begin{array}{ccccc} L^{\text{add}} \mathcal{S}_0(\mathcal{C}) & \xrightarrow{J_0} & L^{\text{add}} \mathcal{F}_0(\mathcal{C}) & \xrightarrow{F_0} & L^{\text{add}} \mathcal{S}_0(\mathcal{C}) \\ E'_m \downarrow & & E_m \downarrow & & \downarrow E'_m \\ L^{\text{add}} \mathcal{S}_m(\mathcal{C}) & \xrightarrow{J_m} & L^{\text{add}} \mathcal{F}_m(\mathcal{C}) & \xrightarrow{F_m} & L^{\text{add}} \mathcal{S}_m(\mathcal{C}) \end{array}$$

of Waldhausen ∞ -categories in which the horizontal composites are equivalences. Since $\mathcal{S}_0(\mathcal{C})$ is a zero object, the left-hand square is a pushout by definition; hence the right-hand square is as well. The geometric realization of the right-hand square is precisely the value of the square (6.8.1) on \mathcal{C} . \square

6.9. **Construction.** Suppose $m \geq 0$ an integer, and suppose $0 \leq k \leq m$. Write $i_k: \mathbf{0} \rightarrow \mathbf{m}$ for the morphism of Δ that carries $0 \in \mathbf{0}$ to $k \in \mathbf{m}$. For any Waldhausen ∞ -category \mathcal{C} , write $I_{m,k}$ for the induced functor $\mathcal{F}_m(\mathcal{C}) \rightarrow \mathcal{F}_0(\mathcal{C})$, and write $I'_{m,k}$ for the induced functor $\mathcal{S}_m(\mathcal{C}) \rightarrow \mathcal{S}_0(\mathcal{C})$.

For future reference, we note that we may now contemplate diagrams $(\Delta^2/\Delta^{\{0,2\}}) \times (\Delta^2/\Delta^{\{0,2\}}) \rightarrow \mathbf{Wald}_\infty$ of the form

$$(6.9.1) \quad \begin{array}{ccccc} \mathcal{S}_0(\mathcal{C}) & \xrightarrow{J_0} & \mathcal{F}_0(\mathcal{C}) & \xrightarrow{F_0} & \mathcal{S}_0(\mathcal{C}) \\ E'_m \downarrow & & E_m \downarrow & & \downarrow E'_m \\ \mathcal{S}_m(\mathcal{C}) & \xrightarrow{J_m} & \mathcal{F}_m(\mathcal{C}) & \xrightarrow{F_m} & \mathcal{S}_m(\mathcal{C}) \\ I'_{m,k} \downarrow & & \downarrow I_{m,k} & & \downarrow I_{m,k} \\ \mathcal{S}_0(\mathcal{C}) & \xrightarrow{J_0} & \mathcal{F}_0(\mathcal{C}) & \xrightarrow{F_0} & \mathcal{S}_0(\mathcal{C}). \end{array}$$

We observe that $\mathcal{F}_0(\mathcal{C}) \simeq \mathcal{C}$ and that $\mathcal{S}_0(\mathcal{C})$ is a zero object. Only the upper right square of (6.9.1) is functorial in \mathbf{m} .

We may now apply the localization functor L^{add} to (6.9.1). In the resulting diagram

$$(6.9.2) \quad \begin{array}{ccccc} L^{\text{add}} \mathcal{S}_0(\mathcal{C}) & \xrightarrow{J_0} & L^{\text{add}} \mathcal{F}_0(\mathcal{C}) & \xrightarrow{F_0} & L^{\text{add}} \mathcal{S}_0(\mathcal{C}) \\ E'_m \downarrow & & E_m \downarrow & & \downarrow E'_m \\ L^{\text{add}} \mathcal{S}_m(\mathcal{C}) & \xrightarrow{J_m} & L^{\text{add}} \mathcal{F}_m(\mathcal{C}) & \xrightarrow{F_m} & L^{\text{add}} \mathcal{S}_m(\mathcal{C}) \\ I'_{m,k} \downarrow & & \downarrow I_{m,k} & & \downarrow I_{m,k} \\ L^{\text{add}} \mathcal{S}_0(\mathcal{C}) & \xrightarrow{J_0} & L^{\text{add}} \mathcal{F}_0(\mathcal{C}) & \xrightarrow{F_0} & L^{\text{add}} \mathcal{S}_0(\mathcal{C}), \end{array}$$

the square in the upper left corner is a pushout, whence every square is a pushout.

We conclude with the observation that the functor $\mathcal{S}: \mathbf{VWald}_\infty \rightarrow \mathbf{VWald}_\infty$ actually takes values in the ∞ -category $\mathbf{V}_{\text{add}} \mathbf{Wald}_\infty$.

6.10. Proposition. *For any virtual Waldhausen ∞ -category \mathcal{X} , the virtual Waldhausen ∞ -category $\mathcal{S}\mathcal{X}$ is distributive.*

Proof. Select a Waldhausen cocartesian fibration $\mathcal{Y} \rightarrow N\Delta^{\text{op}}$ such that \mathcal{X} is equivalent to $|\mathcal{Y}|_{N\Delta^{\text{op}}}$. The claim is that for any compact Waldhausen ∞ -category \mathcal{C} and any integer $m \geq 0$, the map

$$\mathrm{H}(\mathcal{F}_m(\mathcal{C}), (\mathcal{S}\mathcal{Y}/N\Delta^{\text{op}} \times N\Delta^{\text{op}})) \rightarrow \mathrm{H}(\mathcal{C}, (\mathcal{S}\mathcal{Y}/N\Delta^{\text{op}} \times N\Delta^{\text{op}})) \times \mathrm{H}(\mathcal{S}_m(\mathcal{C}), (\mathcal{S}\mathcal{Y}/N\Delta^{\text{op}} \times N\Delta^{\text{op}}))$$

is an equivalence. Since $N\Delta^{\text{op}}$ is sifted and geometric realization commutes with products, we reduce to the case in which \mathcal{Y} is a single Waldhausen ∞ -category.

Let us now use Joyal's ∞ -categorical variant of Quillen's Theorem A [37, Th. 4.1.3.1]. For any object

$$((p, \alpha), (q, \beta)) \in \mathrm{H}(\mathcal{C}, (\mathcal{S}\mathcal{Y}/N\Delta^{\text{op}})) \times \mathrm{H}(\mathcal{S}_m(\mathcal{C}), (\mathcal{S}\mathcal{Y}/N\Delta^{\text{op}})),$$

write $J((p, \alpha), (q, \beta))$ for the pullback

$$\begin{array}{ccc} J((p, \alpha), (q, \beta)) & \longrightarrow & \mathrm{H}(\mathcal{C}, (\mathcal{S}\mathcal{Y}/N\Delta^{\text{op}})) \times \mathrm{H}(\mathcal{S}_m(\mathcal{C}), (\mathcal{S}\mathcal{Y}/N\Delta^{\text{op}})) \\ \downarrow & & \downarrow \\ \mathrm{H}(\mathcal{F}_m(\mathcal{C}), (\mathcal{S}\mathcal{Y}/N\Delta^{\text{op}})) & \longrightarrow & (\mathrm{H}(\mathcal{C}, (\mathcal{S}\mathcal{Y}/N\Delta^{\text{op}})) \times \mathrm{H}(\mathcal{S}_m(\mathcal{C}), (\mathcal{S}\mathcal{Y}/N\Delta^{\text{op}})))_{((p, \alpha), (q, \beta)) /} \end{array}$$

We may identify $J((p, \alpha), (q, \beta))$ with a quasicategory whose objects are tuples $(r, \gamma, \mu, \nu, \sigma, \tau)$, where $r \geq 0$ is an integer, $\gamma: \mathcal{S}_m(\mathcal{C}) \rightarrow \mathcal{S}_r\mathcal{Y}$ is an exact functor, $\mu: [\mathbf{r}] \rightarrow [\mathbf{p}]$ and $\nu: [\mathbf{r}] \rightarrow [\mathbf{q}]$ are morphisms of Δ , and $\sigma: \mu^*\alpha \xrightarrow{\sim} \gamma|_{\mathcal{C}}$ and $\tau: \nu^*\beta \xrightarrow{\sim} \gamma|_{\mathcal{S}_m(\mathcal{C})}$ are equivalences of exact functors.

Denote by κ the constant functor $J((p, \alpha), (q, \beta)) \rightarrow J((p, \alpha), (q, \beta))$ at the object

$$(0, 0, \{0\} \hookrightarrow [\mathbf{p}], \{0\} \hookrightarrow [\mathbf{q}], 0, 0).$$

To prove that $J((p, \alpha), (q, \beta))$ is contractible, we construct an endofunctor λ and natural transformations

$$\mathrm{id} \longleftarrow \lambda \longrightarrow \kappa.$$

For any integer $r \geq 0$, consider $1 + r$ as the category $[\mathbf{r}]^{\triangleleft}$; we define the functor λ by

$$\lambda(r, \gamma, \mu, \nu, \sigma, \tau) := (1 + r, s_0 \circ \gamma, \mu', \nu', \sigma', \tau'),$$

where $\mu'|_{[\mathbf{r}]} = \mu$ and $\mu'(-\infty) = 0$, $\nu'|_{[\mathbf{r}]} = \nu$ and $\nu'(-\infty) = 0$, and σ' and τ' are the obvious extensions of σ and τ . The inclusion $r \hookrightarrow 1 + r$ induces a natural transformation $\lambda \rightarrow \mathrm{id}$, and the inclusion $\{-\infty\} \hookrightarrow 1 + r$ induces a natural transformation $\lambda \rightarrow \kappa$. \square

We thus have the following enhancement of Pr. 6.8.

6.10.1. Corollary. *The diagram*

$$\begin{array}{ccc} \mathbf{VWald}_\infty & & \\ \downarrow L^{\text{add}} & \searrow \mathcal{S} & \\ & & \mathbf{V}_{\text{add}} \mathbf{Wald}_\infty \\ & \nearrow \Sigma_{\mathbf{V}_{\text{add}} \mathbf{Wald}_\infty} & \\ \mathbf{V}_{\text{add}} \mathbf{Wald}_\infty & & \end{array}$$

commutes (up to homotopy), where $\Sigma_{\mathbf{V}_{\text{add}} \mathbf{Wald}_\infty}$ is the suspension endofunctor on $\mathbf{V}_{\text{add}} \mathbf{Wald}_\infty$.

7. ADDITIVE THEORIES

We introduce additive functors $\mathbf{Wald}_\infty \rightarrow \mathcal{E}_*$. We show that additive functors $\mathbf{Wald}_\infty \rightarrow \mathcal{E}_*$ may be identified with excisive functors $\mathbf{V}_{\text{add}}\mathbf{Wald}_\infty \rightarrow \mathcal{E}_*$ that preserve sifted colimits. Since suspension in this ∞ -category is given by the functor \mathcal{S} , the best excisive approximation can be constructed by looping the functor composed with \mathcal{S} .

7.1. Definition. Suppose \mathcal{E} an ∞ -topos. By an \mathcal{E} -valued theory, we shall here mean a pointed, ω -continuous functor $\mathbf{Wald}_\infty \rightarrow \mathcal{E}_*$. We write $\text{Thy}(\mathcal{E})$ for the full subcategory of $\text{Fun}(\mathbf{Wald}_\infty, \mathcal{E}_*)$ spanned by the \mathcal{E} -valued theories.

A theory $\phi \in \text{Thy}(\mathcal{E})$ will be said to be *grouplike* if, for any Waldhausen ∞ -category \mathcal{C} , the *shear functor* $\mathcal{C} \oplus \mathcal{C} \rightarrow \mathcal{C} \oplus \mathcal{C}$ defined by $(X, Y) \mapsto (X, X \vee Y)$ induces an equivalence $\pi_0\phi(\mathcal{C} \oplus \mathcal{C}) \rightarrow \pi_0\phi(\mathcal{C} \oplus \mathcal{C})$.

We presently arrive at our main theorem.

7.2. Theorem. Suppose \mathcal{E} an ∞ -topos. Suppose ϕ an \mathcal{E} -valued theory. Then the following are equivalent.

(7.2.1) For any Waldhausen ∞ -category \mathcal{C} , any nonnegative integer m , and any integer $0 \leq k \leq m$, the functors

$$\phi(F_m): \phi(\mathcal{F}_m(\mathcal{C})) \rightarrow \phi(\mathcal{S}_m(\mathcal{C})) \quad \text{and} \quad \phi(I_{m,k}): \phi(\mathcal{F}_m(\mathcal{C})) \rightarrow \phi(\mathcal{F}_0(\mathcal{C}))$$

exhibit $\phi(\mathcal{F}_m(\mathcal{C}))$ as a product of $\phi(\mathcal{S}_m(\mathcal{C}))$ and $\phi(\mathcal{F}_0(\mathcal{C}))$.

(7.2.2) For any Waldhausen ∞ -category \mathcal{C} and for any functor $S_*(\mathcal{C}): N\Delta^{\text{op}} \rightarrow \mathbf{Wald}_\infty$ that classifies the Waldhausen cocartesian fibration $\mathcal{S}(\mathcal{C}) \rightarrow N\Delta^{\text{op}}$, the induced functor $\phi \circ S_*(\mathcal{C}): N\Delta^{\text{op}} \rightarrow \mathcal{E}_*$ is a group object [37, Df. 7.2.2.1].

(7.2.3) The theory ϕ is grouplike, and for any Waldhausen ∞ -category \mathcal{C} and any integer m , the functors

$$\phi(F_m): \phi(\mathcal{F}_m(\mathcal{C})) \rightarrow \phi(\mathcal{S}_m(\mathcal{C})) \quad \text{and} \quad \phi(I_{m,0}): \phi(\mathcal{F}_m(\mathcal{C})) \rightarrow \phi(\mathcal{F}_0(\mathcal{C}))$$

exhibit $\phi(\mathcal{F}_m(\mathcal{C}))$ as a product of $\phi(\mathcal{S}_m(\mathcal{C}))$ and $\phi(\mathcal{F}_0(\mathcal{C}))$.

(7.2.4) The theory ϕ is grouplike, and for any Waldhausen ∞ -category \mathcal{C} , the functors

$$\phi(F_1): \phi(\mathcal{F}_1(\mathcal{C})) \rightarrow \phi(\mathcal{S}_1(\mathcal{C})) \quad \text{and} \quad \phi(I_{1,0}): \phi(\mathcal{F}_1(\mathcal{C})) \rightarrow \phi(\mathcal{F}_0(\mathcal{C}))$$

exhibit $\phi(\mathcal{F}_1(\mathcal{C}))$ as a product of $\phi(\mathcal{S}_1(\mathcal{C}))$ and $\phi(\mathcal{F}_0(\mathcal{C}))$.

(7.2.5) The theory ϕ is grouplike, it carries direct sums to products, and, for any Waldhausen ∞ -category \mathcal{C} , the images of $\phi(I_{1,1})$ and $\phi(I_{1,0} \oplus F_1)$ in $\text{Mor}_{h\mathcal{E}_*}(\mathcal{F}_1(\mathcal{C}), \mathcal{C})$ are equal.

(7.2.6) The theory ϕ is grouplike, and for any Waldhausen ∞ -category \mathcal{C} , any nonnegative integer m , and any functor $S_*(\mathcal{C}): N\Delta^{\text{op}} \rightarrow \mathbf{Wald}_\infty$ that classifies the Waldhausen cocartesian fibration $\mathcal{S}(\mathcal{C}) \rightarrow N\Delta^{\text{op}}$, the induced functor $\phi \circ S_*(\mathcal{C}): N\Delta^{\text{op}} \rightarrow \mathcal{E}_*$ is a category object [38, Df. 1.1.1]; that is, the maps $\{i-1, i\} \hookrightarrow \mathbf{m}$ of Δ induce morphisms that exhibit $\phi(\mathcal{S}_{\mathbf{m}}(\mathcal{C}))$ as a product of the objects $\phi(\mathcal{S}_{\{i-1, i\}}(\mathcal{C}))$ for $i \in \{1, 2, \dots, m\}$.

(7.2.7) The left derived functor $\Phi: \mathbf{VWald}_\infty \rightarrow \mathcal{E}_*$ of ϕ factors through an excisive functor

$$\Phi_{\text{add}}: \mathbf{V}_{\text{add}}\mathbf{Wald}_\infty \rightarrow \mathcal{E}_*.$$

Proof. The equivalence of conditions (7.2.1) and (7.2.2) follows from Pr. 5.12 and the proof of [37, Pr. 6.1.2.6]. (Also see [37, Rk. 6.1.2.8].) Conditions (7.2.3) and (7.2.6) are clearly special cases of (7.2.1) and (7.2.2), respectively, and condition (7.2.4) is a special case of (7.2.3). The equivalence of (7.2.3) and (7.2.6) also follows directly from Pr. 5.12.

Let us show that (7.2.4) implies (7.2.5). We begin by noting that we have an analogue of the commutative diagram (6.3.1):

$$\begin{array}{ccccc} \phi(\mathcal{C} \oplus \mathcal{D}) & \longrightarrow & \phi(\mathcal{F}_1(\mathcal{C} \oplus \mathcal{D})) & \longrightarrow & \phi(\mathcal{C} \oplus \mathcal{D}) \\ \downarrow & & \downarrow & & \downarrow \\ \phi(\mathcal{C}) \times \phi(\mathcal{D}) & \longrightarrow & \phi(\mathcal{C} \oplus \mathcal{D}) \times \phi(\mathcal{C} \oplus \mathcal{D}) & \longrightarrow & \phi(\mathcal{C}) \times \phi(\mathcal{D}), \end{array}$$

and once again it is a retract diagram in \mathcal{E}_* . Since \mathcal{E}_* admits filtered colimits, equivalences therein are closed under retracts, so since the center vertical morphism is an equivalence, the outer vertical morphisms are as well. Hence ϕ carries direct sums to products. Now the exact functor $I_{1,0} \oplus F_1$ admits a (homotopy) section

$\sigma: \mathcal{C} \oplus \mathcal{C} \longrightarrow \mathcal{F}_1(\mathcal{C})$ such that $I_{1,1} \circ \sigma \simeq \nabla$. Hence if ϕ satisfies (7.2.4), then $\phi(I_{1,0} \oplus F_1)$ is an equivalence with homotopy inverse $\phi(\sigma)$, whence $\phi(I_{1,1})$ and $\phi(I_{1,0} \oplus F_1)$ are equal in $\text{Mor}_{h\mathcal{E}_*}(\mathcal{F}_1(\mathcal{C}), \mathcal{C})$.

It is now easy to see that (7.2.6) implies (7.2.2).

We now show that (7.2.5) implies (7.2.3). For any natural number m , suppose the images of $\phi(I_{1,1})$ and $\phi(I_{1,0} \oplus F_1)$ in $\text{Mor}_{h\mathcal{E}_*}(\mathcal{F}_1(\mathcal{F}_m(\mathcal{C})), \mathcal{F}_m(\mathcal{C}))$ are equal; we must show that $\phi(I_{m,0} \oplus F_m)$ is an equivalence. Compose $I_{1,1}$ and $I_{1,0} \oplus F_1$ with the exact functor $\mathcal{F}_m(\mathcal{C}) \longrightarrow \mathcal{F}_1(\mathcal{F}_m(\mathcal{C}))$ that sends a filtered object

$$X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots \longrightarrow X_m$$

to the cofibration of filtered objects given by the diagram

$$\begin{array}{ccccccc} X_0 & = & X_0 & = & X_0 & = & \cdots = X_0 \\ \parallel & & \downarrow & & \downarrow & & \downarrow \\ X_0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & \cdots \longrightarrow X_m \end{array}$$

the exact functor $I_{m,0} \oplus F_m$ also admits a (homotopy) section $\sigma: \mathcal{C} \oplus \mathcal{S}_m(\mathcal{C}) \longrightarrow \mathcal{F}_m(\mathcal{C})$ such that $I_{m,1} \circ \sigma \simeq \nabla$, and applying our condition on ϕ , we find that $\phi(\sigma \circ (I_{m,0} \oplus F_m)) \simeq \phi(\text{id})$.

We now set about showing that (7.2.3) implies (7.2.7). First, we show that Φ factors through a functor

$$\Phi_{\text{add}}: \mathbf{V}_{\text{add}} \mathbf{Wald}_{\infty} \longrightarrow \mathcal{E}_*.$$

As above, we find that Φ carries direct sums to products, and from this we deduce that Φ carries morphisms of the class S described in Pr. 6.7 to equivalences. We further claim that the family T of those morphisms of \mathbf{VWald}_{∞} that are carried to equivalences by Φ is a strongly saturated class. Since Φ sends direct sums to products, it carries any finite coproduct of elements of T to equivalences. Moreover, since Φ preserves sifted colimits, it preserves any morphism that can be exhibited as a small sifted colimit of elements of T . Hence the full subcategory of $\mathcal{O}(\mathbf{VWald}_{\infty})$ spanned by the elements of T is closed under all small colimits. Finally, to prove that any pushout $\psi': \mathcal{X}' \longrightarrow \mathcal{Y}'$ of an element $\psi: \mathcal{X} \longrightarrow \mathcal{Y}$ of T (along any morphism $\mathcal{X} \longrightarrow \mathcal{X}'$), we note that we may exhibit ψ' as the natural morphism of geometric realizations¹

$$|B_*(\mathcal{X}', \mathcal{X}, \mathcal{X})| \longrightarrow |B_*(\mathcal{X}', \mathcal{X}, \mathcal{Y})|,$$

where the simplicial objects $B_*(\mathcal{X}', \mathcal{X}, \mathcal{X})$ and $B_*(\mathcal{X}', \mathcal{X}, \mathcal{Y})$ are two-sided bar constructions defined by

$$B_n(\mathcal{X}', \mathcal{X}, \mathcal{X}) := \mathcal{X}' \oplus \mathcal{X}^{\oplus n} \oplus \mathcal{X} \quad \text{and} \quad B_n(\mathcal{X}', \mathcal{X}, \mathcal{Y}) := \mathcal{X}' \oplus \mathcal{X}^{\oplus n} \oplus \mathcal{Y}.$$

Since T is closed under formation of products, each map $B_n(\mathcal{X}', \mathcal{X}, \mathcal{X}) \longrightarrow B_n(\mathcal{X}', \mathcal{X}, \mathcal{Y})$ is an element of T , and since T is closed under geometric realizations, the morphism $\mathcal{X}' \longrightarrow \mathcal{Y}'$ is an element of T . Hence T is strongly saturated and therefore contains \bar{S} ; thus Φ factors through a functor $\Phi_{\text{add}}: \mathbf{V}_{\text{add}} \mathbf{Wald}_{\infty} \longrightarrow \mathcal{E}_*$.

We now show that Φ_{add} is excisive. For any nonnegative integer m , apply ϕ to the diagram (6.9.1) with $k = 0$. The lower right corner of the resulting diagram is a pullback. Hence the upper right corner of the diagram resulting from applying ϕ to the diagram (6.9.1) is also a pullback. Now we may form the geometric realization of this simplicial diagram of squares to obtain a square

$$\begin{array}{ccc} \Phi(\mathcal{F}_0(\mathcal{C})) & \longrightarrow & \Phi(\mathcal{S}_0(\mathcal{C})) \\ \downarrow & & \downarrow \\ \Phi(\mathcal{F}(\mathcal{C})) & \longrightarrow & \Phi(\mathcal{S}(\mathcal{C})). \end{array}$$

It follows from the Segal delooping machine ([47] and [37, Lm. 7.2.2.11]) that this square is a pullback as well, since for any functor $S_*(\mathcal{C}): N\Delta^{\text{op}} \longrightarrow \mathbf{Wald}_{\infty}$ classified by the Waldhausen cocartesian fibration $\mathcal{S}(\mathcal{C}) \longrightarrow N\Delta^{\text{op}}$, the simplicial object $\Phi \circ S_*(\mathcal{C})$ is a group object, and $\mathcal{F}(\mathcal{C})$ and $\mathcal{S}_0(\mathcal{C})$ are zero objects. Since \mathcal{S} is a suspension functor in $\mathbf{V}_{\text{add}} \mathbf{Wald}_{\infty}$, we find that the natural transformation $\Phi_{\text{add}} \longrightarrow \Omega_{\mathcal{E}} \circ \Phi_{\text{add}} \circ \Sigma$ is an equivalence, whence F_{add} is excisive [41, Pr. 1.4.2.13].

To complete the proof, it remains to show that (7.2.7) implies (7.2.1). It follows from (7.2.7) that for any nonnegative integer m and any integer $0 \leq k \leq m$, applying Φ to (6.9.1) yields the same result as applying

¹We are grateful to Jacob Lurie for this observation.

Φ_{add} to (6.9.2). Since the lower right square of the latter diagram is a pushout in $\mathbf{V}_{\text{add}}\mathbf{Wald}_{\infty}$, the excisive functor F_{add} carries it to a pullback square in \mathcal{E}_* , whence we obtain the first condition. \square

7.3. Definition. Suppose \mathcal{E} an ∞ -topos. An \mathcal{E} -valued theory ϕ will be said to be *additive* just in case it satisfies any of the equivalent conditions of the previous theorem. We denote by $\text{Add}(\mathcal{E})$ the full subcategory of $\text{Thy}(\mathcal{E})$ spanned by the additive theories.

Now our main theorem (Th. 7.2) yields an identification of additive theories and excisive functors on distributive virtual Waldhausen ∞ -categories.

7.4. Theorem. *Suppose \mathcal{E} an ∞ -topos. The functor $L^{\text{add}} \circ j$ induces an equivalence of ∞ -categories*

$$\text{Exc}_{\mathcal{G}}(\mathbf{V}_{\text{add}}\mathbf{Wald}_{\infty}, \mathcal{E}_*) \xrightarrow{\sim} \text{Add}(\mathcal{E}),$$

where $\text{Exc}_{\mathcal{G}}(\mathbf{V}_{\text{add}}\mathbf{Wald}_{\infty}, \mathcal{E}_*) \subset \text{Fun}^*(\mathbf{V}_{\text{add}}\mathbf{Wald}_{\infty}, \mathcal{E}_*)$ is the full subcategory spanned by excisive functors that preserve small sifted simplicial sets.

Proof. It follows from Th. 7.2 that composition with $L^{\text{add}} \circ j$ defines an essentially surjective functor

$$\text{Exc}_{\mathcal{G}}(\mathbf{V}_{\text{add}}\mathbf{Wald}_{\infty}, \mathcal{E}_*) \longrightarrow \text{Add}(\mathbf{Wald}_{\infty}, \mathcal{E}_*).$$

To see that this functor is fully faithful, it suffices to note that we have a commutative diagram

$$\begin{array}{ccc} \text{Exc}_{\mathcal{G}}(\mathbf{V}_{\text{add}}\mathbf{Wald}_{\infty}, \mathcal{E}_*) & \longrightarrow & \text{Add}(\mathbf{Wald}_{\infty}, \mathcal{E}_*) \\ \downarrow & & \downarrow \\ \text{Fun}(\mathbf{V}_{\text{add}}\mathbf{Wald}_{\infty}, \mathcal{E}_*) & \hookrightarrow & \text{Fun}(\mathbf{VWald}_{\infty}, \mathcal{E}_*) \end{array}$$

in which the vertical functors are fully faithful by definition, and the bottom functor is fully faithful, since the ∞ -category $\mathbf{V}_{\text{add}}\mathbf{Wald}_{\infty}$ is a localization of \mathbf{VWald}_{∞} . \square

By virtue of [41, Pr. 1.4.4.10], this result now yields a *canonical* delooping of any additive functor.

7.4.1. Corollary. *Suppose \mathcal{X} an ∞ -topos. Then composition with the canonical functor $\Omega^{\infty}: \text{Stab}(\mathcal{E}) \rightarrow \mathcal{E}_*$ induces an equivalence of ∞ -categories*

$$\text{Fun}_{\mathcal{G}}^{\text{rex}}(\mathbf{V}_{\text{add}}\mathbf{Wald}_{\infty}, \text{Stab}(\mathcal{E})) \longrightarrow \text{Add}(\mathcal{E}_*),$$

where $\text{Fun}_{\mathcal{G}}^{\text{rex}}(\mathbf{V}_{\text{add}}\mathbf{Wald}_{\infty}, \text{Stab}(\mathcal{E})) \subset \text{Fun}(\mathbf{V}_{\text{add}}\mathbf{Wald}_{\infty}, \text{Stab}(\mathcal{E}))$ denotes the full subcategory spanned by the right exact functors $\Phi: \mathbf{V}_{\text{add}}\mathbf{Wald}_{\infty} \rightarrow \text{Stab}(\mathcal{E})$ such that $\Omega^{\infty} \circ \Phi: \mathbf{V}_{\text{add}}\mathbf{Wald}_{\infty} \rightarrow \mathcal{E}_*$ preserves sifted colimits.

We now find that any theory admits an additive approximation given by a Goodwillie differential. The nature of colimits computed in $\mathbf{V}_{\text{add}}\mathbf{Wald}_{\infty}$ will then permit us to describe this additive approximation as an ∞ -categorical S_{\bullet} construction. As a result, we find that any such theory deloops to a *connective* spectrum.

We first need the following well-known lemma, which follows from [41, Lm. 5.3.6.17] or, alternately, from a suitable generalization of [41, Cor. 5.1.3.7].

7.5. Lemma. *For any ∞ -topos \mathcal{E} , the loop functor $\Omega_{\mathcal{E}}: \mathcal{E}_* \rightarrow \mathcal{E}_*$ preserves sifted colimits of connected objects.*

7.6. Theorem. *Suppose \mathcal{E} an ∞ -topos. The inclusion functor*

$$\text{Add}(\mathcal{E}) \hookrightarrow \text{Thy}(\mathcal{E})$$

admits a left adjoint D given by the Goodwillie differential [23, 25, 26]

$$D\phi \simeq \text{colim}_{n \rightarrow \infty} \Omega_{\mathcal{E}}^n \circ \Phi \circ \mathcal{S}^n \circ j,$$

where $\Phi: \mathbf{VWald}_{\infty} \rightarrow \mathcal{E}_*$ is the left derived functor of ϕ .

Proof. By [26, Th. 1.8] or [41, Cor. 7.1.1.10], the inclusion $\mathrm{Exc}_{\mathcal{F}}(\mathbf{V}_{\mathrm{add}} \mathbf{Wald}_{\infty}, \mathcal{E}_*) \hookrightarrow \mathrm{Fun}_{\mathcal{F}}^*(\mathbf{V}_{\mathrm{add}} \mathbf{Wald}_{\infty}, \mathcal{E}_*)$ admits a left adjoint given by the assignment

$$\Phi \longmapsto \mathrm{colim}_{n \rightarrow \infty} \Omega_{\mathcal{E}}^n \circ \Phi \circ \Sigma_{\mathbf{V}_{\mathrm{add}} \mathbf{Wald}_{\infty}}^n.$$

Now the inclusion $i: \mathbf{V}_{\mathrm{add}} \mathbf{Wald}_{\infty} \hookrightarrow \mathbf{VWald}_{\infty}$ induces a left adjoint

$$\mathrm{Fun}_{\mathcal{F}}^*(\mathbf{VWald}_{\infty}, \mathcal{E}_*) \longrightarrow \mathrm{Fun}_{\mathcal{F}}^*(\mathbf{V}_{\mathrm{add}} \mathbf{Wald}_{\infty}, \mathcal{E}_*)$$

to the forgetful functor induced by L^{add} . By composing these adjoints, we thus obtain a left adjoint \mathbf{D} to the forgetful functor

$$\mathrm{Exc}_{\mathcal{F}}(\mathbf{V}_{\mathrm{add}} \mathbf{Wald}_{\infty}, \mathcal{E}_*) \hookrightarrow \mathrm{Fun}_{\mathcal{F}}^*(\mathbf{VWald}_{\infty}, \mathcal{E}_*).$$

The left adjoint \mathbf{D} is given by the assignment

$$\Phi \longmapsto \mathrm{colim}_{n \rightarrow \infty} \Omega_{\mathcal{E}}^n \circ \Phi \circ i \circ \Sigma_{\mathbf{V}_{\mathrm{add}} \mathbf{Wald}_{\infty}}^n.$$

By Cor. 6.10.1, if $n \geq 1$, then one may rewrite the functor $\Omega_{\mathcal{E}}^n \circ F \circ i \circ \Sigma_{\mathbf{V}_{\mathrm{add}} \mathbf{Wald}_{\infty}}^n$ as

$$\Omega_{\mathcal{E}}^n \circ \Phi \circ i \circ \Sigma_{\mathbf{V}_{\mathrm{add}} \mathbf{Wald}_{\infty}}^n \circ L^{\mathrm{add}} \circ i \simeq \Omega_{\mathcal{E}}^n \circ \Phi \circ i \circ \mathcal{S}^n.$$

Now if $\Phi: \mathbf{VWald}_{\infty} \rightarrow \mathcal{E}_*$ is the left derived functor of a theory, then for any virtual Waldhausen ∞ -category \mathcal{Y} , since Φ is pointed, and since $\mathcal{S}(\mathcal{Y})$ is the colimit of a simplicial virtual Waldhausen ∞ -category $S_*(\mathcal{Y})$ with $S_0(\mathcal{Y}) \simeq 0$, the object $\Phi(\mathcal{S}(\mathcal{Y}))$ is connected as well. By Lm 7.5, $\Omega_{\mathcal{E}}$ commutes with sifted colimits of connected objects of \mathcal{E} , whence it follows that the composite functor

$$\mathrm{Thy}(\mathcal{E}) \xrightarrow{\sim} \mathrm{Fun}_{\mathcal{G}}^*(\mathbf{VWald}_{\infty}, \mathcal{E}_*) \hookrightarrow \mathrm{Fun}_{\mathcal{F}}^*(\mathbf{VWald}_{\infty}, \mathcal{E}_*) \longrightarrow \mathrm{Exc}_{\mathcal{F}}(\mathbf{V}_{\mathrm{add}} \mathbf{Wald}_{\infty}, \mathcal{E}_*)$$

in fact factors through the full subcategory $\mathrm{Exc}_{\mathcal{G}}(\mathbf{V}_{\mathrm{add}} \mathbf{Wald}_{\infty}, \mathcal{E}_*) \subset \mathrm{Exc}_{\mathcal{F}}(\mathbf{V}_{\mathrm{add}} \mathbf{Wald}_{\infty}, \mathcal{E}_*)$. Thanks to Th. 7.4, the functor \mathbf{D} consequently descends to a functor $D: \mathrm{Thy}(\mathcal{E}) \rightarrow \mathrm{Add}(\mathcal{E})$ given by the assignment

$$\Phi \longmapsto \mathrm{colim}_{n \rightarrow \infty} \Omega_{\mathcal{E}}^n \circ \Phi \circ \Sigma_{\mathbf{V}_{\mathrm{add}} \mathbf{Wald}_{\infty}}^n \circ L^{\mathrm{add}} \circ j.$$

Now another application of Cor. 6.10.1 completes the proof. \square

7.7. Definition. The left adjoint

$$D: \mathrm{Thy}(\mathcal{E}) \longrightarrow \mathrm{Add}(\mathcal{E})$$

of the previous corollary will be called the *additivization*.

Suppose $\phi: \mathbf{Wald}_{\infty} \rightarrow \mathcal{E}_*$ a theory; denote by F its left derived functor. For any virtual Waldhausen ∞ -category \mathcal{Y} and any natural number n , since the virtual Waldhausen ∞ -category $\mathcal{S}^n(\mathcal{Y})$ is the colimit of a reduced n -simplicial diagram $S_*(S_*(\cdots S_*(\mathcal{Y}) \cdots))$, it follows that the object $F(\mathcal{S}^n(\mathcal{Y}))$ is n -connected. This proves the following.

7.8. Proposition. *The canonical delooping [Cor. 7.4.1] of the additivization $D\phi$ of a theory $\phi: \mathbf{Wald}_{\infty} \rightarrow \mathcal{E}_*$ is valued in connective spectra:*

$$\mathbf{Wald}_{\infty} \longrightarrow \mathrm{Stab}(\mathcal{E})_{\geq 0}.$$

Now we study a particularly well-behaved class of theories.

7.9. Definition. Suppose \mathcal{E} an ∞ -topos. Then a theory $\phi \in \mathrm{Thy}(\mathcal{E})$ is said to be *pre-additive* if it carries direct sums of Waldhausen ∞ -categories to products in \mathcal{E} .

As we shall see, this property is enjoyed by many of the most interesting theories.

7.10. Proposition. *Suppose \mathcal{E} an ∞ -topos, and suppose $\phi \in \mathrm{Thy}(\mathcal{E})$ a pre-additive theory with left derived functor Φ . Then the morphisms*

$$\Phi(\mathcal{S}(\mathcal{F}_m(\mathcal{C}))) \longrightarrow \Phi(\mathcal{S}(\mathcal{C})) \quad \text{and} \quad \Phi(\mathcal{S}(\mathcal{F}_m(\mathcal{C}))) \longrightarrow \Phi(\mathcal{S}(\mathcal{S}_m(\mathcal{C})))$$

induced by $I_{m,0}$ and F_m together exhibit $\Phi(\mathcal{S}(\mathcal{F}_m(\mathcal{C})))$ as a product of $\Phi(\mathcal{S}(\mathcal{C}))$ and $\Phi(\mathcal{S}(\mathcal{S}_m(\mathcal{C})))$.

Proof. Since ϕ is pre-additive, the morphism from $\Phi(\mathcal{S}(\mathcal{F}_m(\mathcal{C})))$ to the desired product may be identified with the morphism

$$\Phi(\mathcal{S}(\mathcal{F}_m(\mathcal{C}))) \longrightarrow \Phi(\mathcal{S}(\mathcal{C}) \oplus \mathcal{S}(\mathcal{S}_m(\mathcal{C}))),$$

which can in turn be identified with the natural morphism

$$\Phi(i \circ \Sigma_{\mathbf{V}_{\text{add}} \mathbf{Wald}_{\infty}} \circ L^{\text{add}}(\mathcal{F}_m(\mathcal{C}))) \longrightarrow \Phi(i \circ \Sigma_{\mathbf{V}_{\text{add}} \mathbf{Wald}_{\infty}} \circ L^{\text{add}}(\mathcal{C} \oplus \mathcal{S}_m(\mathcal{C})))$$

by Cor. 6.10.1. The upper right corner of (6.9.2) is a pushout, and since $E_m \oplus J_m$ is a section of $I_{m,0} \oplus F_m$, the natural morphism $L^{\text{add}}(\mathcal{F}_m(\mathcal{C})) \longrightarrow L^{\text{add}}(\mathcal{C} \oplus \mathcal{S}_m(\mathcal{C}))$ is an equivalence. \square

By Th. 7.2, we obtain the following repackaging of Waldhausen's Additivity Theorem.

7.10.1. Corollary. *Suppose \mathcal{E} an ∞ -topos, and suppose $\phi \in \text{Thy}(\mathcal{E})$ a pre-additive theory with left derived functor Φ . Then the additivization is given by*

$$D\phi \simeq \Omega \circ \Phi \circ \mathcal{S} \circ j.$$

Suppose \mathcal{E} an ∞ -topos, and suppose $\phi \in \text{Thy}(\mathcal{E})$ a pre-additive theory. Then the counit $\phi \longrightarrow D\phi$ is the initial object of the ∞ -category $\text{Add}(\mathcal{E}) \times_{\text{Thy}(\mathcal{E})} \text{Thy}(\mathcal{E})_{\phi/}$. By Th. 7.2, this means that $D\phi$ is the initial object of the full subcategory of $\text{Thy}(\mathcal{E})_{\phi/}$ spanned by those natural transformations $\phi \longrightarrow \phi'$ such that for any Waldhausen ∞ -category \mathcal{C} and for any functor $S_*(\mathcal{C}): N\Delta^{\text{op}} \longrightarrow \mathbf{Wald}_{\infty}$ classified by the Waldhausen cocartesian fibration $\mathcal{S}(\mathcal{C}) \longrightarrow N\Delta^{\text{op}}$, the induced functor $\phi' \circ S_*(\mathcal{C}): N\Delta^{\text{op}} \longrightarrow \mathcal{E}_*$ is a group object. Motivated by this, we may now note that the inclusion of the full subcategory $\mathbf{Grp}(\mathcal{E})$ of $\text{Fun}(N\Delta^{\text{op}}, \mathcal{E})$ spanned by the group objects admits a left adjoint L . (It is an straightforward matter to note that $\mathbf{Grp}(\mathcal{E}) \subset \text{Fun}(N\Delta^{\text{op}}, \mathcal{E})$ is stable under arbitrary limits and filtered colimits; alternatively, one may find a small set S of morphisms of $\text{Fun}(N\Delta^{\text{op}}, \mathcal{E})$ such that a simplicial object X of \mathcal{E} is a group object if and only if X is S -local.) Hence one may consider the following composite functor L_*^{ϕ} :

$$\mathbf{Wald}_{\infty} \xrightarrow{S_*} \text{Fun}(N\Delta^{\text{op}}, \mathbf{Wald}_{\infty}) \xrightarrow{\phi} \text{Fun}(N\Delta^{\text{op}}, \mathcal{E}) \xrightarrow{L} \mathbf{Grp}(\mathcal{E}).$$

If $\text{ev}_1: \mathbf{Grp}(\mathcal{E}) \longrightarrow \mathcal{E}_*$ is the functor given by evaluation at 1, then the functor $\text{ev}_1 \circ L$ may be identified with the functor $\Omega_{\mathcal{E}} \circ \text{colim}_{N\Delta^{\text{op}}}$. It therefore follows from the previous corollary that the resulting functor L_1^{ϕ} can be identified with the additivization of ϕ . This provides us with a *local* recognition principle for $D\phi$.

7.11. Proposition. *Suppose \mathcal{E} an ∞ -topos, and suppose $\phi \in \text{Thy}(\mathcal{E})$ a pre-additive theory. Then for any Waldhausen ∞ -category \mathcal{C} , the functor $S_*(\mathcal{C}): N\Delta^{\text{op}} \longrightarrow \mathbf{Wald}_{\infty}$ classified by the Waldhausen cocartesian fibration $\mathcal{S}(\mathcal{C}) \longrightarrow N\Delta^{\text{op}}$, the object $D\phi(\mathcal{C})$ is canonically equivalent to underlying object of the group object that is initial in the ∞ -category $\mathbf{Grp}(\mathcal{E}) \times_{\text{Fun}(N\Delta^{\text{op}}, \mathcal{E})} \text{Fun}(N\Delta^{\text{op}}, \mathcal{E})_{\phi \circ S_*(\mathcal{C})/}$.*

7.12. One may hope to study the rest of the Taylor tower of a theory. In particular, for any positive integer n and any theory $\phi \in \text{Thy}(\mathcal{E})$, one may define a symmetric “multi-additive” theory $D^{(n)}\phi$ via a formula

$$D^{(n)}\phi(\mathcal{C}_1, \dots, \mathcal{C}_n) = \text{colim}_{(j_1, \dots, j_n)} \Omega_{\mathcal{E}}^{j_1 + \dots + j_n} \text{cr}_n \Phi(\mathcal{S}^{j_1} \mathcal{C}_1, \dots, \mathcal{S}^{j_n} \mathcal{C}_n),$$

where Φ is the left derived functor of ϕ , and $\text{cr}_n \Phi$ is the n -th cross-effect functor of the restriction of Φ to $\mathbf{V}_{\text{add}} \mathbf{Wald}_{\infty}$. However, if ϕ is pre-additive, then for $n \geq 2$, the cross-effect functor $\text{cr}_n \Phi$ vanishes, whence $D^{(n)}\phi$ vanishes as well. As a result, the Taylor tower for Φ is constant above the first level. More informally, the best polynomial approximation to Φ is linear. Consequently, if $\phi: \mathbf{Wald}_{\infty} \longrightarrow \mathcal{E}_*$ is pre-additive, then Φ factors through an n -excisive functor $\mathbf{V}_{\text{add}} \mathbf{Wald}_{\infty} \longrightarrow \mathcal{E}_*$ for some $n \geq 1$ if and only if ϕ is an additive theory, in which case n may be allowed to be 1. This seems to suggest a rather peculiar dichotomy: a pre-additive theory is either additive or staunchly non-analytic.

8. EASY CONSEQUENCES OF ADDITIVITY

Additivity is a relatively heavy constraint on a theory, and from our work in the previous section, we can immediately deduce the following classical results.

8.1. Proposition (Eilenberg swindle). *Suppose \mathcal{E} an ∞ -topos, and suppose $\phi \in \text{Add}(\mathcal{E})$. Then for any Waldhausen ∞ -category \mathcal{C} that admits countable coproducts, $\phi(\mathcal{C})$ is terminal in \mathcal{E} .*

Proof. Denote by I the set of natural numbers, regarded as a discrete ∞ -category, and denote by $\psi: \mathcal{C} \rightarrow \mathcal{C}$ the composite of the constant functor $\mathcal{C} \rightarrow \text{Fun}(I, \mathcal{C})$ followed by its left adjoint $\text{Fun}(I, \mathcal{C}) \rightarrow \mathcal{C}$. The inclusion $\{0\} \hookrightarrow I$ and the successor bijection $\sigma: I \xrightarrow{\sim} I - \{0\}$ together specify a natural cofibration $\text{id} \rightarrow \psi$. This defines an exact functor $\mathcal{C} \rightarrow \mathcal{F}_1(\mathcal{C})$. Applying $I_{1,1}$ and $I_{1,0} \oplus F_1$ to this functor, we find that $\phi(\psi) = \phi(\text{id}) + \phi(\psi)$, whence $\phi(\text{id}) = 0$. \square

8.2. Proposition (Suspension). *Suppose \mathcal{A} a Waldhausen ∞ -category whose pair structure is maximal. Then for any additive theory $\phi \in \text{Add}(\mathcal{E})$, the suspension functor $\Sigma: \mathcal{A} \rightarrow \mathcal{A}$ induces multiplication by -1 on the group object $\phi(\mathcal{A})$.*

Proof. This follows directly from the existence of the pushout square of endofunctors of \mathcal{A}

$$\begin{array}{ccc} \text{id} & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \Sigma. \end{array}$$

\square

8.2.1. Corollary. *Suppose \mathcal{A} a Waldhausen ∞ -category whose pair structure is maximal. Write $\widetilde{\mathbf{Sp}}(\mathcal{A})$ for the colimit*

$$\mathcal{A} \xrightarrow{\Sigma} \mathcal{A} \xrightarrow{\Sigma} \dots \xrightarrow{\Sigma} \mathcal{A} \xrightarrow{\Sigma} \dots$$

in \mathbf{Wald}_∞ . Then for any additive theory $\phi \in \text{Add}(\mathcal{E})$, the canonical functor

$$\Sigma^\infty: \mathcal{A} \rightarrow \widetilde{\mathbf{Sp}}(\mathcal{A})$$

induces an equivalence $\phi(\mathcal{A}) \rightarrow \phi(\widetilde{\mathbf{Sp}}(\mathcal{A}))$.

The ∞ -category $\widetilde{\mathbf{Sp}}(\mathcal{A})$ is generally not the stabilization of \mathcal{A} , but it is frequently closely related to it. For example, if \mathcal{A} is the ∞ -category of pointed Kan complexes that are *finite* in the sense that they are equivalent to simplicial sets with finitely many nondegenerate simplices, then $\widetilde{\mathbf{Sp}}(\mathcal{A})$ is the ∞ -category of finite spectra. Similarly, we have the following.

8.3. Proposition. *Suppose \mathcal{A} a Waldhausen ∞ -category whose pair structure is maximal. Then $\widetilde{\mathbf{Sp}}(\mathcal{A})$ is equivalent to $\mathbf{Sp}(\text{Ind}(\mathcal{A}))^\omega$.*

Proof. The colimit of the sequence

$$\mathcal{A} \xrightarrow{\Sigma} \mathcal{A} \xrightarrow{\Sigma} \dots \xrightarrow{\Sigma} \mathcal{A} \xrightarrow{\Sigma} \dots$$

in \mathbf{Wald}_∞ agrees with the same colimit taken in $\mathbf{Cat}_\infty(\kappa_1)^{\text{Rex}}$ by [37, Pr. 5.5.7.11] and Pr. 4.4. Since Ind is a left adjoint [37, Pr. 5.5.7.10], the colimit of the sequence

$$\text{Ind } \mathcal{A} \xrightarrow{\Sigma} \text{Ind } \mathcal{A} \xrightarrow{\Sigma} \dots \xrightarrow{\Sigma} \text{Ind } \mathcal{A} \xrightarrow{\Sigma} \dots$$

in $\mathbf{Pr}_\omega^{\text{L}}$ is $\text{Ind}(\widetilde{\mathbf{Sp}}(\mathcal{A}))$. By [37, Nt. 5.5.7.7], there is an equivalence between $\mathbf{Pr}_\omega^{\text{L}}$ and $(\mathbf{Pr}_\omega^{\text{R}})^{\text{op}}$, whence $\text{Ind}(\widetilde{\mathbf{Sp}}(\mathcal{A}))$ can be identified with the limit of the sequence

$$\dots \xrightarrow{\Omega} \text{Ind } \mathcal{A} \xrightarrow{\Omega} \dots \xrightarrow{\Omega} \text{Ind } \mathcal{A} \xrightarrow{\Omega} \text{Ind } \mathcal{A}$$

in $\mathbf{Pr}_\omega^{\text{R}}$. Since the inclusion $\mathbf{Pr}_\omega^{\text{R}} \hookrightarrow \mathbf{Cat}_\infty(\kappa_1)$ preserves limits [37, Pr. 5.5.7.6], it follows that $\text{Ind}(\widetilde{\mathbf{Sp}}(\mathcal{A})) \simeq \mathbf{Sp}(\text{Ind}(\mathcal{A}))$. Now the functor $C \rightarrow C^\omega$ is an equivalence of ∞ -categories between $\mathbf{Pr}_\omega^{\text{R}}$ and the full subcategory of $\mathbf{Cat}_\infty(\kappa_1)^{\text{Lex}}$ spanned by the essentially small, idempotent complete ∞ -categories, and since the limit of idempotent complete ∞ -categories is idempotent complete, it follows that $\widetilde{\mathbf{Sp}}(\mathcal{A}) \simeq \text{Ind}(\widetilde{\mathbf{Sp}}(\mathcal{A}))^\omega \simeq \mathbf{Sp}(\text{Ind}(\mathcal{A}))^\omega$. \square

By analyzing the additivization of the Yoneda embedding, we now find that a every distributive virtual Waldhausen ∞ -category one step away from being an infinite loop object. This implies that the ∞ -category $\mathbf{V}_{\text{add}} \mathbf{Wald}_\infty$ can be said to admit a much stronger form of the Blakers–Massey excision theorem than the ∞ -category of spaces. Armed with this, we give an easy necessary and sufficient criterion for a morphism of virtual Waldhausen ∞ -categories to induce an equivalence on every additive theory.

8.4. Definition. We shall call a theory $\phi \in \text{Thy}(\mathcal{E})$ *left exact* just in case its left derived functor Φ preserves finite limits.

Clearly every left exact theory is pre-additive. Moreover, the best excisive approximation $P_1(G \circ F)$ to the composite $G \circ F$ of a suitable functor $F: C \rightarrow D$ with a functor $G: D \rightarrow D'$ that preserves finite limits is simply the composite $G \circ P_1(F)$. Accordingly, we have the following.

8.5. Lemma. *Suppose $\phi \in \text{Thy}(\mathcal{E})$ a left exact theory. Then $D\phi \simeq \Phi \circ \Omega_{\mathbf{V}_{\text{add}} \mathbf{Wald}_{\infty}} \circ \mathcal{S}$.*

8.6. Example. The Yoneda embedding $y: \mathbf{Wald}_{\infty} \rightarrow \mathcal{P}(\mathbf{Wald}_{\infty}^{\omega})$ preserves finite limits; hence it factors through a left exact theory

$$y: \mathbf{Wald}_{\infty} \rightarrow \mathcal{P}_*(\mathbf{Wald}_{\infty}^{\omega})$$

and its left derived functor $Y: \mathbf{VWald}_{\infty} \hookrightarrow \mathcal{P}_*(\mathbf{Wald}_{\infty}^{\omega})$, which is simply the canonical inclusion. Consequently, thanks to Cor. 7.10.1, the additivization of y is now given by the formula

$$Dy \simeq \Omega \circ \mathcal{S} \circ j.$$

Let us give some equivalent descriptions of the functor Dy . Since $\mathcal{F}(\mathcal{C})$ is contractible, one may write

$$Dy(\mathcal{C}) \simeq \mathcal{F}(\mathcal{C}) \times_{\mathcal{S}(\mathcal{C})} \mathcal{F}(\mathcal{C}).$$

Alternately, since suspension in $\mathbf{V}_{\text{add}} \mathbf{Wald}_{\infty}$ is given by \mathcal{S} , the functor $Dy(\mathcal{C}): \mathbf{Wald}_{\infty}^{\omega, \text{op}} \rightarrow \mathbf{Kan}_*$ can be described by the formula

$$Dy(\mathcal{C})(\mathcal{D}) \simeq \text{Map}_{\mathbf{VWald}_{\infty}}(\mathcal{S}\mathcal{D}, \mathcal{S}\mathcal{C}).$$

In other words, $\Omega_{\mathbf{V}_{\text{add}} \mathbf{Wald}_{\infty}} \Sigma_{\mathbf{V}_{\text{add}} \mathbf{Wald}_{\infty}}$ is the Goodwillie derivative of the identity on $\mathbf{V}_{\text{add}} \mathbf{Wald}$, and $\Omega \mathcal{S}$ is the Goodwillie derivative of L^{add} . We may regard this as a very strong form of Blakers–Massey excision.

More generally, we can attempt to study the circumstances under which a sequence of virtual Waldhausen ∞ -categories gives rise to a fiber sequence under any additive functor. In this direction we have Pr. 8.9 below, which is an analogue of Waldhausen’s [57, Pr. 1.5.5 and Cor. 1.5.7].

8.7. Notation. Suppose $\psi: \mathcal{B} \rightarrow \mathcal{A}$ an exact functor of Waldhausen ∞ -categories. Write $\mathcal{K}(\psi)$ for the realization of the Waldhausen cocartesian fibration $\mathcal{F}\mathcal{A} \times_{\mathcal{S}\mathcal{A}} \mathcal{S}\mathcal{B} \rightarrow N\Delta^{\text{op}}$.

8.8. Given an exact functor $\psi: \mathcal{B} \rightarrow \mathcal{A}$, the virtual Waldhausen ∞ -category $\mathcal{K}(\psi)$ is the geometric realization of the simplicial Waldhausen ∞ -category whose m -simplices consist of a totally filtered object

$$0 \rightarrowtail U_1 \rightarrowtail U_2 \rightarrowtail \dots \rightarrowtail U_m$$

of \mathcal{B} , a filtered object

$$X_0 \rightarrowtail X_1 \rightarrowtail X_2 \rightarrowtail \dots \rightarrowtail X_m$$

of \mathcal{A} , and a diagram

$$\begin{array}{ccccccc} X_0 & \rightarrowtail & X_1 & \rightarrowtail & X_2 & \rightarrowtail & \dots \rightarrowtail X_m \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrowtail & \psi(U_1) & \rightarrowtail & \psi(U_2) & \rightarrowtail & \dots \rightarrowtail \psi(U_m) \end{array}$$

of \mathcal{A} in which every square is a pushout.

The object $\mathcal{K}(\psi)$ is not itself the corresponding fiber product of virtual Waldhausen ∞ -categories; however, for any additive functor $\phi: \mathbf{Wald}_{\infty} \rightarrow \mathcal{E}_*$ with left derived functor Φ , we shall now show that $\Phi(\mathcal{K}(\psi))$ is the fiber of $\Phi(\mathcal{S}\mathcal{A}) \rightarrow \Phi(\mathcal{S}\mathcal{B})$.

8.9. Theorem (Generic Fibration Theorem I). *Suppose $\psi: \mathcal{B} \rightarrow \mathcal{A}$ an exact functor of Waldhausen ∞ -categories. Then for any additive theory $\phi: \mathbf{Wald}_{\infty} \rightarrow \mathcal{E}_*$ with left derived functor Φ , there is a diagram*

$$\begin{array}{ccccc} \phi(\mathcal{B}) & \longrightarrow & \phi(\mathcal{A}) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \Phi(\mathcal{K}(\psi)) & \longrightarrow & \Phi(\mathcal{S}(\mathcal{B})). \end{array}$$

of \mathcal{E}_* in which each square is a pullback.

Proof. For any vertex $\mathbf{m} \in N\Delta^{\text{op}}$, there exist functors

$$s := (E_m \oplus \mathcal{I}_m(\psi), \text{pr}_2): \mathcal{F}_0(\mathcal{A}) \oplus \mathcal{I}_m(\mathcal{B}) \longrightarrow \mathcal{F}_m(\mathcal{A}) \times_{\mathcal{I}_m(\mathcal{A})} \mathcal{I}_m(\mathcal{B})$$

and

$$p := (I_{m,0} \circ \text{pr}_1) \oplus \text{pr}_2: \mathcal{F}_m(\mathcal{A}) \times_{\mathcal{I}_m(\mathcal{A})} \mathcal{I}_m(\mathcal{B}) \longrightarrow \mathcal{F}_0(\mathcal{A}) \oplus \mathcal{I}_m(\mathcal{B}).$$

Clearly $p \circ s \simeq \text{id}$; we claim that $\phi(s \circ p) \simeq \phi(\text{id})$ in \mathcal{E}_* . This follows from additivity applied to the functor

$$\mathcal{F}_m(\mathcal{A}) \times_{\mathcal{I}_m(\mathcal{A})} \mathcal{I}_m(\mathcal{B}) \longrightarrow \mathcal{F}_1(\mathcal{F}_m(\mathcal{A}) \times_{\mathcal{I}_m(\mathcal{A})} \mathcal{I}_m(\mathcal{B}))$$

given by the cofibration of functors $(E_m \circ I_{m,0} \circ \text{pr}_1, 0) \xrightarrow{\sim} \text{id}$. Thus $\phi(\mathcal{F}_m(\mathcal{A}) \times_{\mathcal{I}_m(\mathcal{A})} \mathcal{I}_m(\mathcal{B}))$ is exhibited as the product $\phi(\mathcal{F}_0(\mathcal{A})) \times \phi(\mathcal{I}_m(\mathcal{B}))$.

Thus we may consider the following commutative diagram of \mathcal{E}_* :

$$\begin{array}{ccccc} \phi(\mathcal{F}_0(\mathcal{B})) & \xrightarrow{\mathcal{F}_0(\psi)} & \phi(\mathcal{F}_0(\mathcal{A})) & \xrightarrow{F_0} & \phi(\mathcal{I}_0(\mathcal{B})) \\ E_m \downarrow & & \downarrow & & \downarrow E'_m \\ \phi(\mathcal{F}_m(\mathcal{B})) & \xrightarrow{(0, F_m)} & \phi(\mathcal{F}_m(\mathcal{A}) \times_{\mathcal{I}_m(\mathcal{A})} \mathcal{I}_m(\mathcal{B})) & \xrightarrow{\text{pr}_1} & \phi(\mathcal{I}_m(\mathcal{B})) \\ & & \downarrow \text{pr}_2 & & \downarrow \mathcal{I}_m(\psi) \\ & & \phi(\mathcal{F}_m(\mathcal{A})) & \xrightarrow{F_m} & \phi(\mathcal{I}_m(\mathcal{A})) \\ I_{m,0} \downarrow & & \downarrow & & \downarrow I'_{m,0} \\ \phi(\mathcal{F}_0(\mathcal{A})) & \xrightarrow{F_0} & \phi(\mathcal{I}_0(\mathcal{A})) & & \end{array}$$

The lower right-hand square is a pullback square by additivity; hence, in light of the identification above, all the squares on the right hand side are pullbacks as well. Again by additivity the wide rectangle of the top row is carried to a pullback square under ϕ , whence all the squares of this diagram are carried to pullback squares.

Since ϕ is additive, so is $\Phi \circ \mathcal{I}$. Hence we obtain a commutative diagram in \mathcal{E}_* :

$$\begin{array}{ccccc} \Phi(\mathcal{I}\mathcal{F}_0(\mathcal{B})) & \longrightarrow & \Phi(\mathcal{I}\mathcal{F}_0(\mathcal{A})) & \longrightarrow & \Phi(\mathcal{I}\mathcal{I}_0(\mathcal{B})) \\ \downarrow & & \downarrow & & \downarrow \\ \Phi(\mathcal{I}\mathcal{F}_m(\mathcal{B})) & \longrightarrow & \Phi(\mathcal{I}\mathcal{F}_m(\mathcal{A}) \times_{\mathcal{I}_m(\mathcal{A})} \mathcal{I}_m(\mathcal{B})) & \longrightarrow & \Phi(\mathcal{I}\mathcal{I}_m(\mathcal{B})) \\ & & \downarrow & & \downarrow \\ & & \Phi(\mathcal{I}\mathcal{F}_m(\mathcal{A})) & \longrightarrow & \Phi(\mathcal{I}\mathcal{I}_m(\mathcal{A})), \end{array}$$

in which every square is a pullback. All the squares in this diagram are functorial in \mathbf{m} , and since the objects that appear are all connected, it follows from [41, Lm. 5.3.6.17] that the squares of the colimit diagram

$$\begin{array}{ccccc} \Phi(\mathcal{I}\mathcal{F}_0(\mathcal{B})) & \longrightarrow & \Phi(\mathcal{I}\mathcal{F}_0(\mathcal{A})) & \longrightarrow & \Phi(\mathcal{I}\mathcal{I}_0(\mathcal{B})) \\ \downarrow & & \downarrow & & \downarrow \\ \Phi(\mathcal{I}\mathcal{F}\mathcal{B}) & \longrightarrow & \Phi(\mathcal{I}\mathcal{K}(\psi)) & \longrightarrow & \Phi(\mathcal{I}\mathcal{I}\mathcal{B}) \\ & & \downarrow & & \downarrow \\ & & \Phi(\mathcal{I}\mathcal{F}\mathcal{A}) & \longrightarrow & \Phi(\mathcal{I}\mathcal{I}\mathcal{A}), \end{array}$$

are all pullbacks. Applying the loop space functor $\Omega_{\mathcal{E}}$ to this diagram now produces a diagram equivalent to the diagram

$$\begin{array}{ccccc} \phi(\mathcal{F}_0(\mathcal{B})) & \longrightarrow & \phi(\mathcal{F}_0(\mathcal{A})) & \longrightarrow & \phi(\mathcal{I}_0(\mathcal{B})) \\ \downarrow & & \downarrow & & \downarrow \\ \Phi(\mathcal{F}\mathcal{B}) & \longrightarrow & \Phi(\mathcal{K}(\psi)) & \longrightarrow & \Phi(\mathcal{I}\mathcal{B}) \\ & & \downarrow & & \downarrow \\ & & \Phi(\mathcal{F}\mathcal{A}) & \longrightarrow & \Phi(\mathcal{I}\mathcal{A}), \end{array}$$

in which every square again is a pullback. \square

8.10. Proposition. *The following are equivalent for an exact functor $\psi: \mathcal{B} \rightarrow \mathcal{A}$ of Waldhausen ∞ -categories.*

- (8.10.1) *For any ∞ -topos \mathcal{E} and any $\phi \in \text{Add}(\mathcal{E})$ with left derived functor $\Phi: \mathbf{Wald}_\infty \rightarrow \mathcal{E}_*$, the induced morphism $\Phi(\psi): \Phi(\mathcal{B}) \rightarrow \Phi(\mathcal{A})$ is an equivalence of \mathcal{E}_* .*
- (8.10.2) *For any ∞ -topos \mathcal{E} and any $\phi \in \text{Add}(\mathcal{E})$ with left derived functor $\Phi: \mathbf{Wald}_\infty \rightarrow \mathcal{E}_*$, the object $\Phi(\mathcal{K}(\psi))$ is contractible.*
- (8.10.3) *The virtual Waldhausen ∞ -category $\mathcal{S}\mathcal{K}(\psi)$ is contractible.*

Proof. In light of Pr. 8.5, if (8.10.1) holds, then the induced morphism $\Omega\mathcal{S}(\psi): \Omega\mathcal{S}(\mathcal{B}) \rightarrow \Omega\mathcal{S}(\mathcal{A})$ is an equivalence of virtual Waldhausen ∞ -categories. Since $\mathcal{S}(\mathcal{B})$ and $\mathcal{S}(\mathcal{A})$ are connected objects of $\mathcal{P}_*(\mathbf{Wald}_\infty)$, this in turn implies (using, say, [41, Cor. 5.1.3.7]) that the induced morphism of virtual Waldhausen ∞ -categories $\mathcal{S}(\psi): \mathcal{S}(\mathcal{B}) \rightarrow \mathcal{S}(\mathcal{A})$ is an equivalence and therefore by Pr. 8.9 that (8.10.2) holds.

Now if (8.10.2) holds, then in particular, $\Omega\mathcal{S}\mathcal{K}(\psi)$ is contractible. Since $\mathcal{S}\mathcal{K}(\psi)$ is connected, it is contractible, yielding (8.10.3).

That the last condition implies the first now follows immediately from Pr. 8.9. \square

9. LABELED WALDHAUSEN ∞ -CATEGORIES AND WALDHAUSEN'S FIBRATION THEOREM

In analogy with Waldhausen's theory of categories with cofibrations and weak equivalences, we study here Waldhausen ∞ -categories with certain compatible classes of “labeled morphisms.” From these *labeled Waldhausen ∞ -categories*, we obtain an important class of virtual Waldhausen ∞ -categories that are typically not Waldhausen ∞ -categories.

9.1. Definition. Suppose \mathcal{C} a Waldhausen ∞ -category. Then a *gluing diagram* in \mathcal{C} is a functor of pairs

$$X: \mathcal{Q}^2 \times (\Delta^1)^b \rightarrow \mathcal{C}$$

such that the squares $X|(\mathcal{Q}^2 \times \Delta^{\{0\}})$ and $X|(\mathcal{Q}^2 \times \Delta^{\{1\}})$ are pushouts. We may depict such gluing diagrams as cubes

$$\begin{array}{ccccc} & & X_{00} & \xrightarrow{\quad} & X_{10} \\ & \swarrow & \downarrow & \swarrow & \downarrow \\ X_{20} & \xrightarrow{\quad} & X_{\infty 0} & & \\ \downarrow & & \downarrow & & \downarrow \\ & \swarrow & X_{01} & \xrightarrow{\quad} & X_{11} \\ & \downarrow & \downarrow & \swarrow & \\ X_{21} & \xrightarrow{\quad} & X_{\infty 1} & & \end{array}$$

in which the top and bottom faces are pushout squares.

9.2. Definition. A *labeling* of a Waldhausen ∞ -category is a subcategory $w\mathcal{C}$ of \mathcal{C} that contains $\iota\mathcal{C}$ such that for any gluing diagram X of \mathcal{C} in which the morphisms $X_{00} \rightarrow X_{01}$, $X_{10} \rightarrow X_{11}$, and $X_{20} \rightarrow X_{21}$ lie in $w\mathcal{C}$, the morphism $X_{\infty 0} \rightarrow X_{\infty 1}$ lies in $w\mathcal{C}$ as well. In this case, the edges of $w\mathcal{C}$ will be called *labeled edges*, and the pair $(\mathcal{C}, w\mathcal{C})$ is called a *labeled Waldhausen ∞ -category*.

A *labeled exact functor* between two labeled Waldhausen ∞ -categories \mathcal{C} and \mathcal{D} is an exact functor $\mathcal{C} \rightarrow \mathcal{D}$ that carries labeled edges to labeled edges.

Note that a labeled Waldhausen ∞ -category has two pair structures: the cofibrations and the labeled edges.

9.3. Example. If $(C, \text{cof } C, wC)$ is a *category with cofibrations and weak equivalences* in the sense of Waldhausen [57, §1.2], then $(NC, N \text{cof } C, NwC)$ is easily seen to be a labeled Waldhausen ∞ -category.

Suppose $(\mathcal{C}, w\mathcal{C})$ a labeled Waldhausen ∞ -category. For gluing diagrams X of \mathcal{C} in which the edges

$$\begin{array}{ll} X_{00} \rightarrow X_{20}, & X_{00} \rightarrow X_{01}, \\ X_{10} \rightarrow X_{\infty 0}, & X_{10} \rightarrow X_{11} \end{array}$$

are all degenerate, the condition above reduces to a guarantee that pushouts of labeled morphism along cofibrations are labeled. For gluing diagrams X of \mathcal{C} in which the edges

$$\begin{array}{ccc} X_{00} & \longrightarrow & X_{10}, & X_{00} & \longrightarrow & X_{01}, \\ X_{20} & \longrightarrow & X_{\infty 0}, & X_{20} & \longrightarrow & X_{21} \end{array}$$

are all degenerate, the condition above reduces to a guarantee that the pushout of any labeled cofibration along any morphism exists and is again a labeled cofibration.

9.4. Notation. Denote by $\ell\mathbf{Wald}_\infty$ the full subcategory of the fiber product $\mathbf{Wald}_\infty^{\text{cart}} \times_{\mathbf{Cat}_\infty^{\text{cart}}} \mathbf{Pair}_\infty^{\text{cart}}$ spanned by the labeled Waldhausen ∞ -categories.

9.5. Proposition. *The ∞ -category $\ell\mathbf{Wald}_\infty$ is presentable.*

Proof. The inclusion

$$\ell\mathbf{Wald}_\infty \hookrightarrow \mathbf{Wald}_\infty \times_{\mathbf{Cat}_\infty} \mathbf{Pair}_\infty$$

admits a left adjoint, which assigns to any object $(\mathcal{C}, \mathcal{C}_\dagger, w\mathcal{C})$ the labeled Waldhausen ∞ -category $(\mathcal{C}, \mathcal{C}_\dagger, \overline{w}\mathcal{C})$, where $\overline{w}\mathcal{C}$ is the smallest labeling containing $w\mathcal{C}$. It is easy to see that $\ell\mathbf{Wald}_\infty$ is stable under filtered colimits in $\mathbf{Wald}_\infty \times_{\mathbf{Cat}_\infty} \mathbf{Pair}_\infty$; hence $\ell\mathbf{Wald}_\infty$ is an accessible localization of $\mathbf{Wald}_\infty \times_{\mathbf{Cat}_\infty} \mathbf{Pair}_\infty$. Since the latter ∞ -category is locally presentable by [37, Pr. 5.5.7.6], the proof is complete. \square

We will also make use of the following more general notion.

9.6. Definition. Suppose S an ∞ -category, and suppose $\mathcal{X} \rightarrow S$ a Waldhausen cartesian fibration. Then a *labeling of \mathcal{X} over S* is a subcategory $w\mathcal{X}$ of \mathcal{X} that satisfies the following conditions.

- (9.6.1) The functor $\mathcal{X} \rightarrow S$ carries labeled edges to equivalences.
- (9.6.2) For any vertex $s \in S$, the subcategory $w\mathcal{X}_s := w\mathcal{X} \cap \mathcal{X}_s$ of the Waldhausen ∞ -category \mathcal{X}_s is a labeling.
- (9.6.3) For any edge $\eta: s \rightarrow t$ of S , the induced functor $\eta^*: \mathcal{X}_t \rightarrow \mathcal{X}_s$ carries labeled edges to labeled edges.

In this case, $(\mathcal{X}/S, w\mathcal{X})$ is called a *labeled Waldhausen cartesian fibration*.

Suppose $\mathcal{X} \rightarrow S$ a Waldhausen cocartesian fibration. Then a *labeling of \mathcal{X} over S* is a subcategory $w\mathcal{X}$ of \mathcal{X} such that $w\mathcal{X}^{\text{op}}$ is a labeling of the Waldhausen cartesian fibration $\mathcal{X}^{\text{op}} \rightarrow S^{\text{op}}$. In this case, $(\mathcal{X}/S, w\mathcal{X})$ is called a *labeled Waldhausen cocartesian fibration*.

9.7. Notation. Denote by $\ell\mathbf{Wald}_\infty^{\text{cart}}$ (respectively, by $\ell\mathbf{Wald}_\infty^{\text{cocart}}$) the full subcategory of $\mathbf{Wald}_\infty^{\text{cart}} \times_{\mathbf{Cat}_\infty^{\text{cart}}} \mathbf{Pair}_\infty^{\text{cart}}$ (resp., of $\mathbf{Wald}_\infty^{\text{cocart}} \times_{\mathbf{Cat}_\infty^{\text{cocart}}} \mathbf{Pair}_\infty^{\text{cocart}}$) spanned by the labeled Waldhausen cartesian fibrations (resp., by the labeled Waldhausen cocartesian fibrations) $((\mathcal{X}/S), w\mathcal{X})$.

The following is again a consequence of [37, Lm. 6.1.1.1].

9.8. Lemma. *The target functors*

$$\ell\mathbf{Wald}_\infty^{\text{cart}} \rightarrow \mathbf{Cat}_\infty \quad \text{and} \quad \ell\mathbf{Wald}_\infty^{\text{cocart}} \rightarrow \mathbf{Cat}_\infty$$

induced by the inclusion $\{1\} \subset \Delta^1$ are both cartesian fibrations.

9.9. Notation. The fibers of the cartesian fibrations

$$\ell\mathbf{Wald}_\infty^{\text{cart}} \rightarrow \mathbf{Cat}_\infty \quad \text{and} \quad \ell\mathbf{Wald}_\infty^{\text{cocart}} \rightarrow \mathbf{Cat}_\infty$$

over an object $\{S\} \subset \mathbf{Cat}_\infty$ will be denoted $\ell\mathbf{Wald}_{\infty/S}^{\text{cart}}$ and $\ell\mathbf{Wald}_{\infty/S}^{\text{cocart}}$, respectively.

Labelings on a Waldhausen ∞ -category \mathcal{C} induces labelings on $\mathcal{F}(\mathcal{C})$ and $\mathcal{S}(\mathcal{C})$ as well.

9.10. Proposition. *If $\mathcal{X} \rightarrow S$ is a Waldhausen cocartesian fibration, and if $w\mathcal{X} \subset \mathcal{X}$ is a labeling, then $w\mathcal{F}(\mathcal{X}/S) \subset \mathcal{F}(\mathcal{X}/S)$ is a labeling of $\mathcal{F}(\mathcal{X}/S)$ over $S \times N\Delta^{\text{op}}$.*

Proof. This is immediate from the following observations for any object $(\mathbf{m}, s) \in N\Delta^{\text{op}} \times S$. First, the functor $\mathcal{Q}^2 \times (\Delta^1)^b \rightarrow \mathcal{F}_m(\mathcal{X}_s)$ is a gluing diagram only if, for each natural number $i \leq m$, the induced functor

$$\mathcal{Q}^2 \times (\Delta^1)^b \rightarrow \mathcal{F}_{\{i\}}(\mathcal{X}_s)$$

is a gluing diagram, and, second, an edge of $\mathcal{F}_m(\mathcal{X}_s)$ is labeled just in case, for each natural number $i \leq m$, the corresponding edge of $\mathcal{F}_{\{i\}}(\mathcal{X}_s)$ is labeled. \square

9.11. Proposition. *If $\mathcal{X} \rightarrow S$ is a Waldhausen cocartesian fibration, then a labeling $w\mathcal{X}$ of \mathcal{X} over S induces a labeling of $\mathcal{S}(\mathcal{X}/S)$ over $N\Delta^{\text{op}} \times S$ such that, for any natural number m and for any $s \in S$, the subcategory $w\mathcal{S}_{1+m}(\mathcal{X}_s) \subset \mathcal{S}_{1+m}(\mathcal{X}_s)$ coincides with the subcategory $w\mathcal{F}_m(\mathcal{X}_s) \subset \mathcal{F}_m(\mathcal{X}_s)$ under the equivalence $\mathcal{S}_{1+m}(\mathcal{X}_s) \simeq \mathcal{F}_m(\mathcal{X}_s)$.*

Proof. In light of Pr. 9.10, it suffices to show that for any morphism $\phi: \mathbf{m} \rightarrow \mathbf{n}$ of Δ and any vertex $s \in S$, the corresponding functor of pairs $\phi^*: \mathcal{S}_n(\mathcal{X}_s) \rightarrow \mathcal{S}_m(\mathcal{X}_s)$ carries labeled morphisms to labeled morphisms. Indeed, suppose $X \rightarrow X'$ a labeled morphism of $\mathcal{S}_n(\mathcal{X}_s)$. For any integer $i \leq n$, the cube

$$\begin{array}{ccccc} & X_{\phi(0)} & \xrightarrow{\quad} & X_{\phi(i)} & \\ \swarrow & \downarrow & & \swarrow & \downarrow \\ 0 & \xrightarrow{\quad} & (\phi^* X)_i & & \\ \parallel & \downarrow & \downarrow & & \downarrow \\ & X'_{\phi(0)} & \xrightarrow{\quad} & X'_{\phi(i)} & \\ & \swarrow & & \swarrow & \\ 0 & \xrightarrow{\quad} & (\phi^* X')_i & & \end{array}$$

in which 0 denotes a zero object is a gluing diagram. Hence by definition, the morphism $(\phi^* X)_i \rightarrow (\phi^* X')_i$ is labeled as well. \square

In order to describe the virtual Waldhausen ∞ -category attached to a labeled Waldhausen ∞ -category, we recall the following construction, similar to the one that played a pivotal role in §5.

9.12. Construction. Suppose $\mathcal{X} \rightarrow S$ a Waldhausen cocartesian fibration, and suppose $w\mathcal{X} \subset \mathcal{X}$ a labeling thereof. Define the simplicial set $\mathcal{B}(\mathcal{X}/S, w\mathcal{X})$ as the simplicial set over $N\Delta^{\text{op}} \times S$ satisfying the following universal property. We require, for any simplicial set K and any map $\sigma: K \rightarrow N\Delta^{\text{op}} \times S$, a bijection

$$\text{Mor}_{/(N\Delta^{\text{op}} \times S)}(K, \mathcal{B}(\mathcal{X}/S, w\mathcal{X})) \cong \text{Mor}_{s\mathbf{Set}(2)/(S, \iota_S)}((K \times_{N\Delta^{\text{op}}} NM, K \times_{N\Delta^{\text{op}}} (NM)_\dagger), (\mathcal{X}, w\mathcal{X})),$$

functorial in σ , where the category $s\mathbf{Set}(2)$ is the one defined in 3.8. In other words, $\mathcal{B}(\mathcal{X}/S, w\mathcal{X})$ is the simplicial set $\mathcal{F}(\mathcal{X}/S, w\mathcal{X})$, where \mathcal{X} is regarded as a pair with its subcategory of labeled edges.

9.13. Thus for any Waldhausen cocartesian fibration $\mathcal{X} \rightarrow S$, and any labeling $w\mathcal{X} \subset \mathcal{X}$ thereof, an object of $\mathcal{B}(\mathcal{X}/S, w\mathcal{X})$ can be represented as a pair (\mathbf{m}, s, X) consisting of an object $\mathbf{m} \in \Delta$, a vertex $s \in S_0$, and a functor $X: \Delta^m \rightarrow w\mathcal{X}_s$. A morphism $(\phi, \eta, \psi): (\mathbf{n}, s, Y) \rightarrow (\mathbf{m}, t, X)$ is a morphism $\phi: \mathbf{m} \rightarrow \mathbf{n}$ of Δ , an edge $\eta: s \rightarrow t$, and a natural transformation ψ from the composite

$$\Delta^m \xrightarrow{\phi} \Delta^n \xrightarrow{Y} w\mathcal{X}_s \hookrightarrow \mathcal{X}_s \xrightarrow{\eta_*} \mathcal{X}_t$$

to the composite

$$\Delta^m \xrightarrow{X} w\mathcal{X}_t \hookrightarrow \mathcal{X}_t$$

Put differently, the functor $N\Delta^{\text{op}} \times S \rightarrow \mathbf{Cat}_\infty$ classified by $\mathcal{B}(\mathcal{X}/S, w\mathcal{X}) \rightarrow N\Delta^{\text{op}} \times S$ assigns to any object (\mathbf{m}, s) of $N\Delta^{\text{op}} \times S$ the full subcategory $\mathcal{B}(\mathcal{X}_s, w\mathcal{X}_s)_m \subset \text{Fun}(\Delta^m, \mathcal{X}_s)$ spanned by those functors that factor through $w\mathcal{X}_s \subset \mathcal{X}_s$.

It is immediate from 3.9 that $\mathcal{B}(\mathcal{X}/S, w\mathcal{X}) \rightarrow N\Delta^{\text{op}} \times S$ is a cocartesian fibration. We endow the ∞ -category $\mathcal{B}(\mathcal{X}/S, w\mathcal{X})$ with a pair structure such that $\mathcal{B}(\mathcal{X}/S, w\mathcal{X}) \rightarrow N\Delta^{\text{op}} \times S$ is a Waldhausen cocartesian fibration in the following manner.

9.14. For any Waldhausen cocartesian fibration $\mathcal{X} \rightarrow S$ and any labeling $w\mathcal{X} \subset \mathcal{X}$ thereof, we endow the ∞ -category $\mathcal{B}(\mathcal{X}/S, w\mathcal{X})$ with a pair structure in the following manner. We let $\mathcal{B}_\dagger(\mathcal{X}/S, w\mathcal{X})$ be the smallest subcategory containing morphisms of the form $(\text{id}, \text{id}, \psi): (\mathbf{m}, s, Y) \rightarrow (\mathbf{m}, s, X)$, where for any integer $0 \leq k \leq m$, the induced morphism $Y_k \rightarrow X_k$ is ingressive.

9.15. **Lemma.** *For any Waldhausen cocartesian fibration $\mathcal{X} \rightarrow S$ and any labeling $w\mathcal{X} \subset \mathcal{X}$ thereof, the cocartesian fibration $p: \mathcal{B}(\mathcal{X}/S, w\mathcal{X}) \rightarrow N\Delta^{\text{op}} \times S$ is a Waldhausen cocartesian fibration.*

Proof. It is plain to see that p is a pair cocartesian fibration.

Now suppose (\mathbf{m}, s) on object of $N\Delta^{\text{op}} \times S$. Since limits and colimits in $\text{Fun}(\Delta^m, \mathcal{X}_s)$ are computed pointwise, a zero object in $\text{Fun}(\Delta^m, \mathcal{X}_s)$ is an essentially constant functor whose value at any point of Δ^m is a zero object. Since any equivalence of \mathcal{X}_s is contained in $w\mathcal{X}_s$, this zero object is contained in $\mathcal{B}(\mathcal{X}_s, w\mathcal{X}_s)_m$ as well. Again since pushouts in $\text{Fun}(\Delta^m, \mathcal{X}_s)$ are formed objectwise, a pushout square in $\text{Fun}(\Delta^m, \mathcal{X}_s)$ is a functor

$$X: \Delta^1 \times \Delta^1 \times \Delta^{\{0,k\}} \rightarrow \mathcal{X}_s$$

such that for any integer $0 \leq k \leq m$, the restriction $X|(\Delta^1 \times \Delta^1 \times \Delta^{\{0,k\}})$ is a pushout square; now if X is in addition a functor of pairs $\mathcal{Q}^2 \times (\Delta^m)^b \rightarrow \mathcal{X}_s$, then it follows directly from the gluing axiom that if $X|(\{0\} \times \Delta^{\{0,k\}})$, $X|(\{1\} \times \Delta^{\{0,k\}})$, and $X|(\{2\} \times \Delta^{\{0,k\}})$ all factor through $w\mathcal{X}_s \subset \mathcal{X}_s$, then so does $X|(\{\infty\} \times \Delta^{\{0,k\}})$. Hence the fibers $\mathcal{B}(\mathcal{X}_s, w\mathcal{X}_s)_m$ of p are Waldhausen ∞ -categories, and, again using the fact that colimits and limits are computed objectwise, we conclude that p is a Waldhausen cocartesian fibration. \square

For any ∞ -category S , it follows from 3.10 that the assignment $(\mathcal{X}/S, w\mathcal{X}) \mapsto \mathcal{B}(\mathcal{X}/S, w\mathcal{X})$ defines a functor

$$\mathcal{B}: \ell\mathbf{Wald}_{\infty/S}^{\text{cocart}} \rightarrow \mathbf{Wald}_{\infty/(N\Delta^{\text{op}} \times S)}^{\text{cocart}}.$$

By composing with the realization functor, we find a functorial construction of virtual Waldhausen ∞ -categories from labeled Waldhausen ∞ -categories:

9.16. **Notation.** Suppose S a sifted ∞ -category. By a small abuse of notation, we denote also as \mathcal{B} the composite functor

$$\ell\mathbf{Wald}_{\infty/S}^{\text{cocart}} \xrightarrow{\mathcal{B}} \mathbf{Wald}_{\infty/(N\Delta^{\text{op}} \times S)}^{\text{cocart}} \xrightarrow{|\cdot|_{N\Delta^{\text{op}} \times S}} \mathbf{VWald}_{\infty}.$$

9.17. We have remarked (Ex. 2.11) that the nerve of an ordinary *category with cofibrations* in the sense of Waldhausen is a Waldhausen ∞ -category, and (Ex. 9.3) that the nerve of a *category with cofibrations and weak equivalences* in the sense of Waldhausen is a labeled Waldhausen ∞ -category. Consequently, a category $(C, \text{cof } C, wC)$ with cofibrations and weak equivalences gives rise to a virtual Waldhausen ∞ -category $\mathcal{B}(NC, N\text{cof } C, NwC)$.

9.18. **Notation.** Note that the pair cartesian fibration $\pi: NM \rightarrow N\Delta^{\text{op}}$ admits a section σ that assigns to any object $\mathbf{m} \in \Delta$ the pair $(\mathbf{m}, 0) \in M$. For any labeled Waldhausen ∞ -category $(\mathcal{C}, w\mathcal{C})$, this section induces a functor of pairs over $N\Delta^{\text{op}}$

$$\mathcal{B}(\mathcal{C}, w\mathcal{C}) \rightarrow (N\Delta^{\text{op}})^b \times \mathcal{C},$$

which carries any object (\mathbf{m}, X) of $\mathcal{F}(\mathcal{C}, w\mathcal{C})$ to the pair (\mathbf{m}, X_0) and any morphism $(\phi, \psi): (\mathbf{n}, Y) \rightarrow (\mathbf{m}, X)$ to the composite

$$Y_0 \rightarrow Y_{\phi(0)} \xrightarrow{\psi_0} X_0.$$

Consequently, we have a functor $\iota\mathcal{B}(\mathcal{C}, w\mathcal{C}) \rightarrow w\mathcal{C}$

More generally, the section σ induces a map of simplicial sets

$$H(\mathcal{D}, \mathcal{B}(\mathcal{C}, w\mathcal{C})) \rightarrow w\text{Fun}_{\mathbf{Wald}_{\infty}}^b(\mathcal{D}, \mathcal{C}),$$

where $w\text{Fun}_{\mathbf{Wald}_{\infty}}^b(\mathcal{D}, \mathcal{C}) \subset \text{Fun}_{\mathbf{Wald}_{\infty}}^b(\mathcal{D}, \mathcal{C})$ denotes the subcategory containing all exact functors $\mathcal{D} \rightarrow \mathcal{C}$ and those natural transformations that are pointwise labeled.

9.19. **Lemma.** *For any labeled Waldhausen ∞ -category $(\mathcal{C}, w\mathcal{C})$ and any compact Waldhausen ∞ -category \mathcal{D} , the map $H(\mathcal{D}, \mathcal{B}(\mathcal{C}, w\mathcal{C})) \rightarrow w\text{Fun}_{\mathbf{Wald}_{\infty}}^b(\mathcal{D}, \mathcal{C})$ induced by σ is a weak homotopy equivalence.*

Proof A. Using (the dual of) Joyal's ∞ -categorical version of Quillen's Theorem A [37, Th. 4.1.3.1], we are reduced to showing that for any exact functor $X: \mathcal{D} \rightarrow \mathcal{C}$, the simplicial set

$$H(\mathcal{D}, \mathcal{B}(\mathcal{C}, w\mathcal{C})) \times_{w \text{Fun}_{\mathbf{Wald}_\infty}^b(\mathcal{D}, \mathcal{C})} w \text{Fun}_{\mathbf{Wald}_\infty}^b(\mathcal{D}, \mathcal{C})_{X/}$$

is weakly contractible. This simplicial set is the geometric realization of the simplicial space

$$\mathbf{n} \mapsto H_{1+n}(\mathcal{D}, \mathcal{B}(\mathcal{C}, w\mathcal{C})) \times_{w \text{Fun}_{\mathbf{Wald}_\infty}^b(\mathcal{D}, \mathcal{C})} \{X\};$$

in particular, it may be identified with the path space of the fiber of the map

$$H(\mathcal{D}, \mathcal{B}(\mathcal{C}, w\mathcal{C})) \rightarrow w \text{Fun}_{\mathbf{Wald}_\infty}^b(\mathcal{D}, \mathcal{C})$$

over the vertex X . \square

Proof B. Consider the ordinary category $\Delta_{w \text{Fun}_{\mathbf{Wald}_\infty}^b(\mathcal{D}, \mathcal{C})}$ of simplices of $w \text{Fun}_{\mathbf{Wald}_\infty}^b(\mathcal{D}, \mathcal{C})$; corresponding to the natural map $N(\Delta_{w \text{Fun}_{\mathbf{Wald}_\infty}^b(\mathcal{D}, \mathcal{C})}^{\text{op}} \times_{\Delta^{\text{op}}} M_{\dagger}) \rightarrow \text{Fun}_{\mathbf{Wald}_\infty}^b(\mathcal{D}, \mathcal{C})$ is a map

$$N\Delta_{w \text{Fun}_{\mathbf{Wald}_\infty}^b(\mathcal{D}, \mathcal{C})}^{\text{op}} \rightarrow H(\mathcal{D}, \mathcal{B}(\mathcal{C}, w\mathcal{C})).$$

This map identifies $N\Delta_{w \text{Fun}_{\mathbf{Wald}_\infty}^b(\mathcal{D}, \mathcal{C})}^{\text{op}}$ with the simplicial subset of $H(\mathcal{D}, \mathcal{B}(\mathcal{C}, w\mathcal{C}))$ whose simplices correspond to maps $\Delta^n \times_{\Delta^{\text{op}}} M_{\dagger} \rightarrow \text{Fun}_{\mathbf{Wald}_\infty}^b(\mathcal{D}, \mathcal{C})$ that carry cocartesian edges (over Δ^n) to degenerate edges. The composite

$$N\Delta_{w \text{Fun}_{\mathbf{Wald}_\infty}^b(\mathcal{D}, \mathcal{C})}^{\text{op}} \rightarrow H(\mathcal{D}, \mathcal{B}(\mathcal{C}, w\mathcal{C})) \rightarrow w \text{Fun}_{\mathbf{Wald}_\infty}^b(\mathcal{D}, \mathcal{C})$$

is the “initial vertex map,” which is a well-known weak equivalence. A simple argument now shows that the map $N\Delta_{w \text{Fun}_{\mathbf{Wald}_\infty}^b(\mathcal{D}, \mathcal{C})}^{\text{op}} \rightarrow H(\mathcal{D}, \mathcal{B}(\mathcal{C}, w\mathcal{C}))$ is also a weak equivalence. \square

Unfortunately, for a labeled Waldhausen ∞ -category $(\mathcal{C}, w\mathcal{C})$, the functor $\mathcal{B}(\mathcal{C}, w\mathcal{C}) \rightarrow (N\Delta^{\text{op}})^b \times \mathcal{C}$ will typically fail to be a morphism of $\mathbf{Wald}_{\infty/N\Delta^{\text{op}}}^{\text{cocart}}$, because the cocartesian edges of $\mathcal{B}(\mathcal{C}, w\mathcal{C})$ will be carried to labeled edges, but not necessarily to equivalences. Hence one may not regard $\sigma_{(\mathcal{C}, w\mathcal{C})}^*$ as a natural transformation of functors $N\Delta^{\text{op}} \rightarrow \mathbf{Wald}_\infty$. To rectify this, we may formally invert the edges in $w\mathcal{C}$ in the ∞ -categorical sense.

9.20. Lemma. *The inclusion functor $\mathbf{Wald}_\infty \hookrightarrow \ell\mathbf{Wald}_\infty$ defined by the assignment $(\mathcal{C}, \mathcal{C}_{\dagger}) \mapsto (\mathcal{C}, \mathcal{C}_{\dagger}, \iota\mathcal{C})$ admits a left adjoint $\ell\mathbf{Wald}_\infty \rightarrow \mathbf{Wald}_\infty$.*

Proof. The inclusion functor $\mathbf{Wald}_\infty \hookrightarrow \ell\mathbf{Wald}_\infty$ preserves all limits and all filtered colimits. Now the result follows from the Adjoint Functor Theorem [37, Cor. 5.5.2.9] along with Pr. 9.5. \square

Let us denote by $w\mathcal{C}^{-1}\mathcal{C}$ the image of a labeled Waldhausen ∞ -category $(\mathcal{C}, w\mathcal{C})$ under the left adjoint above. The canonical exact functor $\mathcal{C} \rightarrow w\mathcal{C}^{-1}\mathcal{C}$ is initial with the property that it carries labeled edges to equivalences. As an example, let us consider the case of an ordinary category with cofibrations and weak equivalences in the sense of Waldhausen [57, §1.2].

9.21. Proposition. *If (C, C_{\dagger}, wC) is a category with cofibrations and weak equivalences that is a partial model category [3] in the sense that: (1) the weak equivalences satisfy the two-out-of-six axiom [17, 9.1], and (2) the weak equivalences and trivial cofibrations are part of a three-arrow calculus of fractions [17, 11.1], then the Waldhausen ∞ -category $NwC^{-1}NC$ is equivalent to the relative nerve $N(C, wC)$, equipped with the pair structure of Ex. 2.11.*

Proof. We first claim that $N(C, wC)$ is a Waldhausen ∞ -category. Indeed, the image of the zero object of C is again a zero object of $N(C, wC)$ as well as an initial object of $N(C, wC)_{\dagger}$. The ∞ -category $\text{Fun}_{\mathbf{Pair}_\infty}(\Lambda_0\mathcal{Q}^2, N(C, wC))$ is the relative nerve of the full subcategory C^{\ulcorner} of $\text{Fun}(\mathbf{1} \cup^{\{0\}} \mathbf{1}, C)$ spanned by those functors that carry the first arrow $0 \rightarrow 1$ to a cofibration, equipped with the objectwise weak equivalences. Similarly, $\text{Fun}_{\mathbf{Pair}_\infty}(\mathcal{Q}^2, N(C, wC))$ is the relative nerve of the full subcategory C^{\sqcup} of $\text{Fun}(\mathbf{1} \times \mathbf{1}, C)$ spanned by those functors that carry the arrows $(0, 0) \rightarrow (0, 1)$ and $(1, 0) \rightarrow (1, 1)$ each to cofibrations,

equipped with the objectwise weak equivalences. The forgetful functor $U: C^\square \rightarrow C^\ulcorner$ and its left adjoint $F: C^\ulcorner \rightarrow C^\square$ are each relative functors, whence they descend to an adjunction

$$F: \mathrm{Ho}(C^\ulcorner) \rightleftarrows \mathrm{Ho}(C^\square): U$$

on the $\mathrm{Ho} \, s\mathbf{Set}$ -enriched homotopy categories, using [17, 36.3]. Thus the forgetful functor

$$\mathrm{Fun}_{\mathbf{Pair}_\infty}(\mathcal{Q}^2, N(C, wC)) \rightarrow \mathrm{Fun}_{\mathbf{Pair}_\infty}(\Lambda_0 \mathcal{Q}^2, N(C, wC))$$

admits a left adjoint. Thus $N(C, wC)$ is a Waldhausen ∞ -category.

Moreover, if $X \twoheadrightarrow Y$ is a cofibration of C and if $X \rightarrow X'$ is an arrow of C , it is easy to see that a square

$$\begin{array}{ccc} X & \twoheadrightarrow & Y \\ \downarrow & & \downarrow \\ X' & \twoheadrightarrow & Y' \end{array}$$

in $N(C, wC)$ is a pushout just in case it is equivalent to the image of a pushout square in C .

Now suppose \mathcal{D} a Waldhausen ∞ -category. Since the canonical functor $NC \rightarrow N(C, wC)$ is exact, there is an induced functor

$$R: \mathrm{Fun}_{\mathbf{Wald}_\infty}^b(N(C, wC), \mathcal{D}) \rightarrow \mathrm{Fun}'_{\mathbf{Wald}_\infty}(C, \mathcal{D}),$$

where $\mathrm{Fun}'_{\mathbf{Wald}_\infty}(C, \mathcal{D}) \subset \mathrm{Fun}_{\mathbf{Wald}_\infty}^b(C, \mathcal{D})$ is the full subcategory spanned by those exact functors that carry arrows in wC to equivalences in \mathcal{D} . The universal property of $N(C, wC)$, combined with the definition of its pair structure, guarantees an equivalence

$$\mathrm{Fun}_{\mathbf{Pair}_\infty}^b(N(C, wC), \mathcal{D}) \xrightarrow{\sim} \mathrm{Fun}'_{\mathbf{Pair}_\infty}(C, \mathcal{D}),$$

where $\mathrm{Fun}'_{\mathbf{Pair}_\infty}(C, \mathcal{D}) \subset \mathrm{Fun}_{\mathbf{Pair}_\infty}^b(C, \mathcal{D})$ is the full subcategory spanned by those functors of pairs that carry arrows in wC to equivalences in \mathcal{D} . Hence R is fully faithful. Since an object (respectively, morphism, square) in $N(C, wC)$ is a zero object (resp., a cofibration, resp., a pushout square along a cofibration) just in case it is equivalent to the image of one under the functor $NC \rightarrow N(C, wC)$, it follows that a functor of pairs $N(C, wC) \rightarrow \mathcal{D}$ that induces an exact functor $C \rightarrow \mathcal{D}$ is itself exact. Thus R is essentially surjective. \square

Let us give another example of a situation in which we can identify the Waldhausen ∞ -category $w\mathcal{C}^{-1}\mathcal{C}$.

9.22. Proposition. *Suppose C a compactly generated ∞ -category containing a zero object, suppose D an accessible localization of C , and suppose the inclusion $D \hookrightarrow C$ preserves filtered colimits. Let $wC^\omega \subset C^\omega$ be the subcategory consisting of the local equivalences, i.e., those morphisms f such that $Lf \simeq 0$. Then $(wC^\omega)^{-1}C^\omega$ is naturally equivalent to D^ω , equipped with its maximal pair structure.*

Proof. Let us show that D^ω has the desired universal property. For any Waldhausen ∞ -category \mathcal{A} , let us write $\mathrm{Fun}'_{\mathbf{Wald}_\infty}(C^\omega, \mathcal{A}) \subset \mathrm{Fun}_{\mathbf{Wald}_\infty}^b(C^\omega, \mathcal{A})$ for the full subcategory spanned by those functors of pairs that carry arrows in wC to equivalences in \mathcal{A} . Consider the subcategory $\mathcal{A}_\dagger \subset \mathcal{A}$. Now \mathcal{A}_\dagger admits all finite colimits, and the inclusion $\mathcal{A}_\dagger \hookrightarrow \mathcal{A}$ induces equivalences

$$\mathrm{Fun}_{\mathbf{Wald}_\infty}(C^\omega, \mathcal{A}) \xrightarrow{\sim} \mathrm{Fun}_{\mathbf{Rex}}(C^\omega, \mathcal{A}_\dagger) \quad \text{and} \quad \mathrm{Fun}'_{\mathbf{Wald}_\infty}(C^\omega, \mathcal{A}) \xrightarrow{\sim} \mathrm{Fun}'_{\mathbf{Rex}}(C^\omega, \mathcal{A}_\dagger),$$

where $\mathrm{Fun}_{\mathbf{Rex}}(C^\omega, \mathcal{A}_\dagger)$ is the ∞ -category of right exact functors, and $\mathrm{Fun}'_{\mathbf{Rex}}(C^\omega, \mathcal{A}_\dagger) \subset \mathrm{Fun}_{\mathbf{Rex}}(C^\omega, \mathcal{A}_\dagger)$ is the full subcategory spanned by those functors that carry arrows in wC to equivalences in \mathcal{A} . We thus aim to show that the functor

$$\mathrm{Fun}_{\mathbf{Rex}}(D^\omega, \mathcal{A}_\dagger) \rightarrow \mathrm{Fun}_{\mathbf{Rex}}(C^\omega, \mathcal{A}_\dagger)$$

is an equivalence.

For any idempotent completion \mathcal{A}'_\dagger of \mathcal{A}_\dagger , the universal property of localizations guarantees that the functor

$$\mathrm{Fun}_{\mathbf{Rex}}(D^\omega, \mathcal{A}'_\dagger) \rightarrow \mathrm{Fun}_{\mathbf{Rex}}(C^\omega, \mathcal{A}'_\dagger)$$

is fully faithful with essential image $\text{Fun}'_{\text{Rex}}(C^\omega, \mathcal{A}'_\dagger)$. Now it is easy to see that the square

$$\begin{array}{ccc} \text{Fun}_{\text{Rex}}(D^\omega, \mathcal{A}_\dagger) & \longrightarrow & \text{Fun}_{\text{Rex}}(D^\omega, \mathcal{A}'_\dagger) \\ \downarrow & & \downarrow \\ \text{Fun}_{\text{Rex}}(C^\omega, \mathcal{A}_\dagger) & \longrightarrow & \text{Fun}_{\text{Rex}}(C^\omega, \mathcal{A}'_\dagger) \end{array}$$

is a pullback square, whence the functor $\text{Fun}_{\text{Rex}}(D^\omega, \mathcal{A}_\dagger) \longrightarrow \text{Fun}_{\text{Rex}}(C^\omega, \mathcal{A}_\dagger)$ is also an equivalence. \square

9.23. Notation. Composing the canonical exact functor $\mathcal{C} \longrightarrow w\mathcal{C}^{-1}\mathcal{C}$ with the functor

$$\mathcal{B}(\mathcal{C}, w\mathcal{C}) \longrightarrow (N\Delta^{\text{op}})^\flat \times \mathcal{C},$$

we obtain a morphism of $\mathbf{Wald}_{\infty/N\Delta^{\text{op}}}^{\text{cocart}}$

$$\mathcal{B}(\mathcal{C}, w\mathcal{C}) \longrightarrow (N\Delta^{\text{op}})^\flat \times w\mathcal{C}^{-1}\mathcal{C}$$

that carries cocartesian edges of $\mathcal{B}(\mathcal{C}, w\mathcal{C})$ to equivalences. Applying $|\cdot|_{N\Delta^{\text{op}}}$, we obtain a morphism of \mathbf{Wald}_∞

$$\gamma_{(\mathcal{C}, w\mathcal{C})}: \mathcal{B}(\mathcal{C}, w\mathcal{C}) \longrightarrow w\mathcal{C}^{-1}\mathcal{C}.$$

9.24. We emphasize that for a general labeled Waldhausen ∞ -category $(\mathcal{C}, w\mathcal{C})$, the comparison morphism $\gamma_{(\mathcal{C}, w\mathcal{C})}$ is not an equivalence; nevertheless, we will find [Pr. 10.8.2] that $\gamma_{(\mathcal{C}, w\mathcal{C})}$ often induces an equivalence on K -theory.

We now aim to prove an analogue of Waldhausen's Generic Fibration Theorem [57, Th. 1.6.4]. For this we require a suitable analogue of Waldhausen's cylinder functor in the ∞ -categorical context. This should reflect the idea that a labeled edge can, to some extent, be replaced by a labeled cofibration.

9.25. Notation. To this end, for any labeled Waldhausen ∞ -category $(\mathcal{A}, \mathcal{A}_\dagger)$, write $w_\dagger \mathcal{A} := w\mathcal{A} \cap \mathcal{A}_\dagger$. The subcategory $w_\dagger \mathcal{A} \subset \mathcal{A}$ defines a new pair structure, but not a new labeling, of \mathcal{A} . Nevertheless, we may consider the full subcategory $\mathcal{B}(\mathcal{A}, w_\dagger \mathcal{A}) \subset \mathcal{F}\mathcal{A}$ spanned by those filtered objects

$$X_0 \twoheadrightarrow X_1 \twoheadrightarrow \cdots \twoheadrightarrow X_m$$

such that each cofibration $X_i \twoheadrightarrow X_{i+1}$ is labeled; we shall regard it as a subpair. One may verify that $\mathcal{B}_m(\mathcal{A}, w_\dagger \mathcal{A}) \subset \mathcal{F}_m(\mathcal{A})$ is a Waldhausen subcategory, and $\mathcal{B}(\mathcal{A}, w_\dagger \mathcal{A}) \longrightarrow N\Delta^{\text{op}}$ is a Waldhausen cocartesian fibration.

For any pair \mathcal{D} , write $w \text{Fun}_{\mathbf{Pair}_\infty}^b(\mathcal{D}, \mathcal{A}) \subset \text{Fun}_{\mathbf{Pair}_\infty}^b(\mathcal{D}, \mathcal{A})$ for the subcategory containing all functors of pairs $\mathcal{D} \longrightarrow \mathcal{C}$ and those natural transformations $\eta: \mathcal{D} \times \Delta^1 \longrightarrow \mathcal{A}$ that are pointwise labeled. Similarly, write $w_\dagger \text{Fun}_{\mathbf{Pair}_\infty}^b(\mathcal{D}, \mathcal{A}) \subset w \text{Fun}_{\mathbf{Pair}_\infty}^b(\mathcal{D}, \mathcal{A})$ for the subcategory containing all functors of pairs $\mathcal{D} \longrightarrow \mathcal{C}$ and those natural transformations $\eta: \mathcal{D} \times \Delta^1 \longrightarrow \mathcal{A}$ that are pointwise labeled cofibrations with the additional property that, for any cofibration $X \twoheadrightarrow Y$ of \mathcal{D} , the resulting square

$$\begin{array}{ccc} \eta(X, 0) & \xrightarrow{\sim} & \eta(X, 1) \\ \downarrow & & \downarrow \\ \eta(Y, 0) & \xrightarrow{\sim} & \eta(Y, 1) \end{array}$$

has the property that the canonical edge from the pushout to $\eta(Y, 1)$ is a cofibration. If \mathcal{D} is a Waldhausen ∞ -category, write $w_\dagger \text{Fun}_{\mathbf{Wald}_\infty}^b(\mathcal{D}, \mathcal{A}) \subset w_\dagger \text{Fun}_{\mathbf{Pair}_\infty}^b(\mathcal{D}, \mathcal{A})$ for the full subcategory spanned by the exact functors.

9.26. Note that the proofs of Lm. 9.19 apply also to the pair $(\mathcal{A}, w_\dagger \mathcal{A})$ to guarantee that for any compact Waldhausen ∞ -category \mathcal{D} , the natural map

$$\text{H}(\mathcal{D}, (\mathcal{B}(\mathcal{A}, w_\dagger \mathcal{A})/N\Delta^{\text{op}})) \longrightarrow w_\dagger \text{Fun}_{\mathbf{Wald}_\infty}^b(\mathcal{D}, \mathcal{A})$$

induced by σ is a weak homotopy equivalence.

9.27. **Definition.** Suppose $(\mathcal{A}, w\mathcal{A})$ a labeled Waldhausen ∞ -category. We shall say that $(\mathcal{A}, w\mathcal{A})$ has *enough cofibrations* if for any small pair of ∞ -categories \mathcal{D} , the inclusion

$$w_{\dagger} \text{Fun}_{\mathbf{Pair}_{\infty}}^b(\mathcal{D}, \mathcal{A}) \hookrightarrow w \text{Fun}_{\mathbf{Pair}_{\infty}}^b(\mathcal{D}, \mathcal{A})$$

is a weak homotopy equivalence.

In particular, if every labeled edge of $(\mathcal{A}, w\mathcal{A})$ is a cofibration, then $(\mathcal{A}, w\mathcal{A})$ has enough cofibrations. More generally, this may prove to be an extremely difficult condition to verify, but the following lemma simplifies matters somewhat.

9.28. **Lemma.** A labeled Waldhausen ∞ -category $(\mathcal{A}, \mathcal{A}_{\dagger})$ has enough cofibrations if and only if, for any of the pairs $\mathcal{D} \in \{\Delta^0, (\Delta^1)^b, (\Delta^1)^{\sharp}\}$, the inclusion

$$w_{\dagger} \text{Fun}_{\mathbf{Pair}_{\infty}}^b(\mathcal{D}, \mathcal{A}) \hookrightarrow w \text{Fun}_{\mathbf{Pair}_{\infty}}^b(\mathcal{D}, \mathcal{A})$$

is a weak homotopy equivalence.

Proof. Any pair of ∞ -categories can be written as a suitable colimit of pairs of the form Δ^0 , $(\Delta^1)^b$, or $(\Delta^1)^{\sharp}$. \square

9.29. **Lemma.** If a labeled Waldhausen ∞ -category $(\mathcal{A}, w\mathcal{A})$ has enough cofibrations, then for any Waldhausen ∞ -category \mathcal{D} , the inclusion $w_{\dagger} \text{Fun}_{\mathbf{Wald}_{\infty}}^b(\mathcal{D}, \mathcal{A}) \hookrightarrow w \text{Fun}_{\mathbf{Wald}_{\infty}}^b(\mathcal{D}, \mathcal{A})$ is a weak homotopy equivalence.

Proof. For any Waldhausen ∞ -category \mathcal{B} , the square

$$\begin{array}{ccc} w_{\dagger} \text{Fun}_{\mathbf{Wald}_{\infty}}^b(\mathcal{D}, \mathcal{A}) & \longrightarrow & w \text{Fun}_{\mathbf{Wald}_{\infty}}^b(\mathcal{D}, \mathcal{A}) \\ \downarrow & & \downarrow \\ w_{\dagger} \text{Fun}_{\mathbf{Pair}_{\infty}}^b(\mathcal{D}, \mathcal{A}) & \longrightarrow & w \text{Fun}_{\mathbf{Pair}_{\infty}}^b(\mathcal{D}, \mathcal{A}) \end{array}$$

is a pullback, and the vertical maps are inclusions of connected components. \square

9.30. **Theorem** (Generic Fibration Theorem II). Suppose $(\mathcal{A}, w\mathcal{A})$ a labeled Waldhausen ∞ -category that has enough cofibrations. Suppose $\phi: \mathbf{Wald}_{\infty} \rightarrow \mathcal{E}_{*}$ an additive theory with left derived functor Φ . Then the inclusion $i: \mathcal{A}^w \hookrightarrow \mathcal{A}$ and the morphism of virtual Waldhausen ∞ -categories $e: \mathcal{A} \rightarrow \mathcal{B}(\mathcal{A}, w\mathcal{A})$ give rise to a fiber sequence

$$\begin{array}{ccc} \phi(\mathcal{A}^w) & \longrightarrow & \phi(\mathcal{A}) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \Phi(\mathcal{B}(\mathcal{A}, w\mathcal{A})). \end{array}$$

Proof. It follows from Pr. 8.9 that it is enough to exhibit an equivalence between $\Phi(\mathcal{B}(\mathcal{A}, w\mathcal{A}))$ and $\Phi(\mathcal{K}(i))$ as objects of $\mathcal{E}_{*, \phi(\mathcal{A})/\cdot}$.

The forgetful functor $\mathcal{K}(i) \rightarrow \mathcal{F}\mathcal{A}$ is fully faithful, and its essential image $\widetilde{\mathcal{F}}^w \mathcal{A}$ consists of those filtered objects

$$X_0 \twoheadrightarrow X_1 \twoheadrightarrow \cdots \twoheadrightarrow X_m$$

such that the induced cofibration $X_i/X_0 \twoheadrightarrow X_{i+1}/X_0$ is labeled; this contains the subcategory $\mathcal{B}(\mathcal{A}, w_{\dagger}\mathcal{A})$.

We claim that for any $m \geq 0$, the induced morphism $\phi(\mathcal{B}_m(\mathcal{A}, w_{\dagger}\mathcal{A})) \rightarrow \phi(\widetilde{\mathcal{F}}_m^w \mathcal{A})$ is an equivalence. Indeed, one may select an exact functor $p: \mathcal{C}_m(i) \rightarrow \mathcal{B}_m(\mathcal{A}, w_{\dagger}\mathcal{A})$ that carries an object

$$\begin{array}{ccccccc} X_0 & \twoheadrightarrow & X_1 & \twoheadrightarrow & X_2 & \twoheadrightarrow & \cdots \twoheadrightarrow X_m \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \twoheadrightarrow & U_1 & \twoheadrightarrow & U_2 & \twoheadrightarrow & \cdots \twoheadrightarrow U_m \end{array}$$

to the filtered object

$$X_0 \twoheadrightarrow X_0 \vee U_1 \twoheadrightarrow X_0 \vee U_2 \twoheadrightarrow \cdots \twoheadrightarrow X_0 \vee U_m.$$

When $m = 0$, this functor is compatible with the canonical equivalences from \mathcal{A} . Additivity now guarantees that p defines a (homotopy) inverse to the morphism $\phi(\mathcal{B}_m(\mathcal{A}, w_{\dagger}\mathcal{A})) \rightarrow \phi(\mathcal{F}_m^w\mathcal{A})$.

Now one has an obvious forgetful functor $\mathcal{B}(\mathcal{A}, w_{\dagger}\mathcal{A}) \rightarrow \mathcal{B}(\mathcal{A}, w\mathcal{A})$ over $N\Delta^{\text{op}}$. We claim that this induces an equivalence of virtual Waldhausen ∞ -categories $|\mathcal{B}(\mathcal{A}, w_{\dagger}\mathcal{A})|_{N\Delta^{\text{op}}} \rightarrow |\mathcal{B}(\mathcal{A}, w\mathcal{A})|_{N\Delta^{\text{op}}}$. So we wish to show that for any compact Waldhausen ∞ -category \mathcal{D} , the morphism

$$\mathrm{H}(\mathcal{D}, (\mathcal{B}(\mathcal{A}, w_{\dagger}\mathcal{A})/N\Delta^{\text{op}})) \rightarrow \mathrm{H}(\mathcal{D}, (\mathcal{B}(\mathcal{A}, w\mathcal{A})/N\Delta^{\text{op}}))$$

of simplicial sets is a weak homotopy equivalence.

By Lm. 9.19 and its extension to the pair $(\mathcal{A}, w_{\dagger}\mathcal{A})$, we have a square

$$\begin{array}{ccc} \mathrm{H}(\mathcal{D}, (\mathcal{B}(\mathcal{A}, w_{\dagger}\mathcal{A})/N\Delta^{\text{op}})) & \longrightarrow & \mathrm{H}(\mathcal{D}, (\mathcal{B}(\mathcal{A}, w\mathcal{A})/N\Delta^{\text{op}})) \\ \downarrow & & \downarrow \\ w_{\dagger}\mathrm{Fun}_{\mathbf{Wald}_{\infty}}^b(\mathcal{D}, \mathcal{C}) & \longrightarrow & w\mathrm{Fun}_{\mathbf{Wald}_{\infty}}^b(\mathcal{D}, \mathcal{C}) \end{array}$$

in which the vertical maps are weak homotopy equivalences. Since $(\mathcal{A}, w\mathcal{A})$ has enough cofibrations, the horizontal map along the bottom is a weak homotopy equivalence as well by Lm. 9.29. \square

Part 3. Algebraic K -theory

We are finally prepared to describe the Waldhausen K -theory of ∞ -categories. We define K -theory as the additivization of the theory ι that assigns to any Waldhausen ∞ -category the maximal Kan complex (or ∞ -groupoid) contained therein. Since the theory ι is representable by the particularly simple Waldhausen ∞ -category $N\Gamma^{\text{op}}$, we obtain, for any additive theory ϕ , a description of the space of natural transformations $K \rightarrow \phi$ as the value of ϕ on $N\Gamma^{\text{op}}$.

Following this, we briefly describe a number of examples that exploit certain features of the algebraic K -theory functor of which we are fond. The first of these relates the algebraic K -theory of the “total space” of a symmetric monoidal ∞ -groupoid to the group completion of a certain *wreath product* of that symmetric monoidal ∞ -groupoid; this can be regarded as a kind of extension of the Barratt–Priddy–Quillen theorem. Next, we extend some of the basic constructions of Waldhausen’s A -theory to the context of ∞ -topoi; there we construct transfers and assembly maps. Following this, we lay the foundations for the algebraic K -theory of E_1 -algebras in a variety of monoidal ∞ -categories, and we prove a straightforward localization theorem. Finally, we extend algebraic K -theory to the context of spectral Deligne–Mumford stacks in the sense of Lurie, and we prove Thomason’s “proto-localization” theorem in this context.

10. THE UNIVERSAL PROPERTY OF WALDHAUSEN K -THEORY

The functor $\iota: \mathbf{Wald}_{\infty} \rightarrow \mathbf{Kan}_{*}$ that assigns to any Waldhausen ∞ -category its interior ∞ -groupoid [Nt. 1.6] is a theory. It is certainly not an additive theory, however, since cofiber sequences in Waldhausen ∞ -categories rarely split. The additivization of ι is (connective) algebraic K -theory.

10.1. Definition. The *algebraic K -theory functor* $K: \mathbf{Wald}_{\infty} \rightarrow \mathbf{Kan}_{*}$ is defined as the additivization $K = D\iota$ of the interior functor $\iota: \mathbf{Wald}_{\infty} \rightarrow \mathbf{Kan}_{*}$. We denote by $\mathbf{K}^{\text{conn}}: \mathbf{Wald}_{\infty} \rightarrow \mathbf{Sp}_{\geq 0}$ its canonical connective delooping, as guaranteed by Cor. 7.4.1 and Pr. 7.8.

Unpacking this definition, we obtain a global universal property of the natural morphism $\iota \rightarrow K$.

10.2. Proposition. *For any additive theory ϕ , the morphism $\iota \rightarrow K$ induces a natural homotopy equivalence*

$$\mathrm{Map}(K, \phi) \xrightarrow{\sim} \mathrm{Map}(\iota, \phi).$$

More informally, K -theory is controlled, as an additive theory, by the theory ι . It is therefore valuable to study this functor in more detail. As a first step, we find that it is corepresentable.

10.3. Notation. For any finite set I , write I_{+} for the finite set $I \sqcup \{\infty\}$. Denote by Γ^{op} the ordinary category of pointed finite sets. Denote by $\Gamma_{\dagger}^{\text{op}} \subset \Gamma^{\text{op}}$ the subcategory comprised of monomorphisms $J_{+} \rightarrow I_{+}$.

10.4. Proposition. *For any Waldhausen ∞ -category \mathcal{C} , the inclusion $\{*\} \hookrightarrow N\Gamma^{\text{op}}$ induces an equivalence of ∞ -categories*

$$\mathbf{Fun}_{\mathbf{Wald}_\infty}(N\Gamma^{\text{op}}, \mathcal{C}) \xrightarrow{\sim} \mathcal{C}.$$

In particular, the functor $\iota: \mathbf{Wald}_\infty \rightarrow \mathbf{Kan}$ is corepresented by the object $N\Gamma^{\text{op}}$.

Proof. Write $N\Gamma_{\leq 1}^{\text{op}}$ for the full subcategory of $N\Gamma^{\text{op}}$ spanned by the objects \emptyset and $*$. Then it follows from Joyal's theorem [37, Pr. 1.2.12.9] that the inclusion $\{*\} \hookrightarrow N\Gamma^{\text{op}}$ induces an equivalence between the full subcategory $\mathbf{Fun}^*(N\Gamma_{\leq 1}^{\text{op}}, \mathcal{C})$ of $\mathbf{Fun}(N\Gamma_{\leq 1}^{\text{op}}, \mathcal{C})$ spanned by functors $z: N\Gamma_{\leq 1}^{\text{op}} \rightarrow \mathcal{C}$ such that $z(\emptyset)$ is a zero object. Now the result follows from the observation that $\mathbf{Fun}_{\mathbf{Wald}_\infty}(N\Gamma^{\text{op}}, \mathcal{C})$ can be identified as the full subcategory of $\mathbf{Fun}(N\Gamma^{\text{op}}, \mathcal{C})$ spanned by those functors $Z: N\Gamma^{\text{op}} \rightarrow \mathcal{C}$ such that (1) $Z(\emptyset)$ is a zero object, and (2) the identity exhibits Z as a left Kan extension of $Z|_{(N\Gamma_{\leq 1}^{\text{op}})}$ along the inclusion $N\Gamma_{\leq 1}^{\text{op}} \hookrightarrow N\Gamma^{\text{op}}$. \square

In the language of Cor. 4.7.3, we find that $W(\Delta^0) \simeq N\Gamma^{\text{op}}$. In any case, from this, the Yoneda lemma combines with Pr. 10.2 to imply the following.

10.4.1. Corollary. *For any additive theory $\phi: \mathbf{Wald}_\infty \rightarrow \mathbf{Kan}_*$, there is a homotopy equivalence*

$$\mathbf{Map}(K, \phi) \simeq \phi(N\Gamma^{\text{op}}),$$

natural in ϕ .

In particular, the Barratt–Priddy–Quillen theorem (cf. 11.17 below) implies the following.

10.4.2. Corollary. *The space of endomorphisms of K -theory is given by*

$$\mathbf{End}(K) \simeq QS^0.$$

Though conceptually pleasant, the universal property of K -theory as an object of $\mathbf{Add}(\mathbf{Kan})$ does not obviously provide an easy recognition principle for the K -theory of any *particular* Waldhausen ∞ -category. For that, we note that ι is pre-additive, and we appeal to Cor. 7.10.1 to obtain the following result.

10.5. Proposition. *For any virtual Waldhausen ∞ -category \mathcal{X} , the K -theory space $K(\mathcal{X})$ is homotopy equivalent to the loop space $\Omega I(\mathcal{S}(\mathcal{X}))$, where I is the left derived functor of ι .*

We observe that for any sifted ∞ -category and any Waldhausen cocartesian fibration $\mathcal{Y} \rightarrow S$, the space $I(\mathcal{S}(|\mathcal{Y}|_S))$ may be computed as the underlying space of the subcategory $\iota_{N\Delta^{\text{op}} \times S} \mathcal{S}(\mathcal{Y})$ of the ∞ -category $\mathcal{S}(\mathcal{Y})$ comprised of the cocartesian edges with respect to the cocartesian fibration $\mathcal{S}(\mathcal{Y}) \rightarrow N\Delta^{\text{op}} \times S$. This provides us with a (singly delooped) model of the algebraic K -theory space $K(\mathcal{Y})$ as the underlying simplicial set of an ∞ -category.

10.5.1. Corollary. *For any sifted ∞ -category S and any Waldhausen cocartesian fibration $\mathcal{Y} \rightarrow S$, the K -theory space $K(|\mathcal{Y}|_S)$ is homotopy equivalent to the loop space $\Omega(\iota_{N\Delta^{\text{op}} \times S} \mathcal{S}(\mathcal{Y}/S))$.*

Since this is precisely how Waldhausen's K -theory is defined [57, §1.3], we obtain a comparison between our ∞ -categorical K -theory and Waldhausen K -theory.

10.5.2. Corollary. *If $(C, \text{cof } C)$ is an ordinary category with cofibrations in the sense of Waldhausen [57, §1.1], then the algebraic K -theory of the Waldhausen ∞ -category $(NC, N(\text{cof } C))$ is naturally equivalent to Waldhausen's algebraic K -theory of $(C, \text{cof } C)$.*

The fact that the algebraic K -theory space $K(\mathcal{X})$ of a virtual Waldhausen ∞ -category \mathcal{X} can be exhibited as the loop space of the underlying simplicial set of an ∞ -category permits us to find the following sufficient condition that a morphism of Waldhausen cocartesian fibrations induce an equivalence on K -theory.

10.5.3. Corollary. *For any sifted ∞ -category S , a morphism $(\mathcal{Y}'/S) \rightarrow (\mathcal{Y}/S)$ of Waldhausen cocartesian fibrations induces an equivalence $K(|\mathcal{Y}'|_S) \xrightarrow{\sim} K(|\mathcal{Y}|_S)$ if the following two conditions are satisfied.*

(10.5.3.1) For any object $X \in \iota_S \mathcal{Y}$, the simplicial set

$$\iota_S \mathcal{Y}' \times_{\iota_S \mathcal{Y}} (\iota_S \mathcal{Y})_{/X}$$

is weakly contractible.

(10.5.3.2) For any object $Y \in \iota_S \mathcal{F}_1(\mathcal{Y}/S)$, the simplicial set

$$\iota_S \mathcal{F}_1(\mathcal{Y}'/S) \times_{\iota_S \mathcal{F}_1(\mathcal{Y}/S)} \iota_S \mathcal{F}_1(\mathcal{Y}/S)_{/Y}$$

is weakly contractible.

Proof. We aim to show that the map $\iota_{N\Delta^{\text{op}} \times S} \mathcal{S}(\mathcal{Y}'/S) \rightarrow \iota_{N\Delta^{\text{op}} \times S} \mathcal{S}(\mathcal{Y}/S)$ is a weak homotopy equivalence; it is enough to show that for any $\mathbf{n} \in \Delta$, the map $\iota_S \mathcal{F}_n(\mathcal{Y}'/S) \rightarrow \iota_S \mathcal{F}_n(\mathcal{Y}/S)$ is a weak homotopy equivalence. Since $\mathcal{F}(\mathcal{Y}'/S)$ and $\mathcal{F}(\mathcal{Y}/S)$ are each category objects [Pr. 5.8], it is enough to prove this claim for $n \in \{0, 1\}$. The result now follows from Joyal's ∞ -categorical version of Quillen's Theorem A [37, Th. 4.1.3.1]. \square

Using Pr. 7.11, we further deduce the following recognition principle for the K -theory of a Waldhausen ∞ -category.

10.6. Proposition. *For any Waldhausen ∞ -category \mathcal{C} , and any functor $S_*(\mathcal{C}): N\Delta^{\text{op}} \rightarrow \mathbf{Wald}_\infty$ classified by the Waldhausen cocartesian fibration $\mathcal{S}(\mathcal{C}) \rightarrow N\Delta^{\text{op}}$, the K -theory space $K(\mathcal{C})$ is the underlying space of the initial object of the ∞ -category $\mathbf{Grp}(\mathbf{Kan}) \times_{\text{Fun}(N\Delta^{\text{op}}, \mathbf{Kan})} \text{Fun}(N\Delta^{\text{op}}, \mathbf{Kan})_{\iota S_* \mathcal{C}/}$.*

We now end this subsection with a discussion of the K -theory of labeled Waldhausen ∞ -categories.

10.7. Notation. Suppose $(\mathcal{C}, w\mathcal{C})$ a labeled Waldhausen ∞ -category [§9]. Then we denote by $K(\mathcal{C}, w\mathcal{C})$ the K -theory space $K(\mathcal{B}(\mathcal{C}, w\mathcal{C}))$.

In light of Lm. 9.19, we now immediately deduce the following.

10.8. Proposition. *For any labeled Waldhausen ∞ -category $(\mathcal{C}, w\mathcal{C})$, the K -theory space $K(\mathcal{C}, w\mathcal{C})$ is weakly homotopy equivalent to the loop space $\Omega(w_{N\Delta^{\text{op}}} \mathcal{S}(\mathcal{C}))$.*

Since this again is precisely how Waldhausen's K -theory is defined [57, §1.3], we obtain a further comparison between our ∞ -categorical K -theory for labeled Waldhausen ∞ -categories and Waldhausen K -theory, analogous to Cor. 10.5.2.

10.8.1. Corollary. *If $(C, \text{cof } C, wC)$ is an ordinary category with cofibrations and weak equivalences in the sense of Waldhausen [57, §1.2], then the algebraic K -theory of the labeled Waldhausen ∞ -category $(NC, N(\text{cof } C), wC)$ is naturally equivalent to Waldhausen's algebraic K -theory of $(C, \text{cof } C, wC)$.*

Using Cor. 10.5.3, we obtain the following.

10.8.2. Corollary. *Suppose $(\mathcal{C}, w\mathcal{C})$ a labeled Waldhausen ∞ -category. Then the comparison morphism $\gamma_{(\mathcal{C}, w\mathcal{C})}$ [Nt. 9.23] induces an equivalence*

$$K(\mathcal{C}, w\mathcal{C}) \rightarrow K(w\mathcal{C}^{-1}\mathcal{C})$$

of K -theory spaces if the following conditions are satisfied.

(10.8.2.1) For any object X of $w\mathcal{C}^{-1}\mathcal{C}$, the simplicial set $w\mathcal{C} \times_{\iota(w\mathcal{C}^{-1}\mathcal{C})} \iota(w\mathcal{C}^{-1}\mathcal{C})_{/X}$ is weakly contractible.

(10.8.2.2) For any object Y of $\mathcal{F}_1(w\mathcal{C}^{-1}\mathcal{C})$, the simplicial set

$$w\mathcal{F}_1(\mathcal{C}) \times_{\iota \mathcal{F}_1(w\mathcal{C}^{-1}\mathcal{C})} \iota \mathcal{F}_1(w\mathcal{C}^{-1}\mathcal{C})_{/Y}$$

is weakly contractible.

Pr. 9.21, combined with Cor. 10.8.2, yields a further corollary.

10.8.3. Corollary. *If (C, wC) is a premodel category in which the weak equivalences and trivial cofibrations are part of a three-arrow calculus of fractions, then the Waldhausen K -theory of $(C, \text{cof } C, wC)$ is canonically equivalent to the K -theory of the relative nerve $N(C, wC)$, equipped with the pair structure described in Ex. 2.11.*

We may also use Cor. 10.8.2 in combination with Pr. 9.22 to specialize the second Generic Fibration Theorem (Th. 9.30) in the following manner.

10.8.4. **Corollary** (Special Fibration Theorem). *Suppose C a compactly generated ∞ -category containing a zero object, suppose D an accessible localization of C , and suppose the inclusion $D \hookrightarrow C$ preserves filtered colimits. Then the functor $L: C^\omega \rightarrow D^\omega$ induces a pullback square*

$$\begin{array}{ccc} K(E^\omega) & \longrightarrow & K(C^\omega) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & K(D^\omega), \end{array}$$

where C^ω and D^ω are equipped with the maximal pair structure, and $E^\omega \subset C^\omega$ is the full subcategory spanned by those objects X such that $LX \simeq 0$.

A yet further specialization of these results is now possible.

10.8.5. **Corollary.** *Suppose C a compactly generated stable ∞ -category equipped with a t -structure such that the inclusion $C_{\leq 0} \hookrightarrow C$ preserves filtered colimits. Then the functor $\tau_{\leq 0}$ induces a pullback square*

$$\begin{array}{ccc} K(C_{\geq 1}^\omega) & \longrightarrow & K(C^\omega) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & K(C_{\leq 0}^\omega), \end{array}$$

where the ∞ -categories that appear are equipped with their maximal pair structure.

11. EXAMPLE: SEGAL'S DELOOPING MACHINE

It will come as no surprise that algebraic K -theory as we have defined it here is closely related to Segal's delooping machine. After all, it was Segal's delooping machine that, in part, inspired Waldhausen's approach to algebraic K -theory. We record a relationship here, using the homotopy theory of symmetric monoidal ∞ -groupoids.

11.1. **Definition.** A *symmetric monoidal ∞ -groupoid* \mathcal{X} is a left fibration $p: \mathcal{X}^\otimes \rightarrow N\Gamma^{\text{op}}$ such that for any pointed finite set I_+ , the maps $\mathcal{X}_I^\otimes \rightarrow \mathcal{X}_{\{i\}}^\otimes$ induced by the inclusions $\{i\}_+ \rightarrow I_+$ exhibit \mathcal{X}_I^\otimes as the product of the Kan complexes $\mathcal{X}_{\{i\}}^\otimes$ for $i \in I$. By a small abuse of notation, we shall write \mathcal{X} for the fiber $\mathcal{X}_{\{*\}}^\otimes$.

Given two symmetric monoidal ∞ -groupoids $p: \mathcal{X}^\otimes \rightarrow N\Gamma^{\text{op}}$ and $q: \mathcal{Y}^\otimes \rightarrow N\Gamma^{\text{op}}$, a *symmetric monoidal functor* $\mathcal{Y} \rightarrow \mathcal{X}$ is a commutative diagram

$$\begin{array}{ccc} \mathcal{Y}^\otimes & \xrightarrow{f} & \mathcal{X}^\otimes \\ & \searrow q & \swarrow p \\ & N\Gamma^{\text{op}} & \end{array}$$

Denote by $\text{Fun}^\otimes(\mathcal{Y}, \mathcal{X})$ the space

$$\text{Fun}^\otimes(\mathcal{Y}, \mathcal{X}) := \text{Fun}(\mathcal{Y}, \mathcal{X}) \times_{\text{Fun}(\mathcal{Y}, N\Gamma^{\text{op}})} \{q\}.$$

One verifies that $\text{Fun}^\otimes(\mathcal{Y}, \mathcal{X})$ is a Kan complex.

Denote by $\mathbf{Gpd}_\infty^{\otimes, \Delta}$ the simplicial category defined in the following manner. The objects of $\mathbf{Gpd}_\infty^{\otimes, \Delta}$ are symmetric monoidal ∞ -groupoids. Given two symmetric monoidal ∞ -groupoids \mathcal{X} and \mathcal{X}' , set

$$\mathbf{Gpd}_\infty^{\otimes, \Delta}(\mathcal{X}', \mathcal{X}) := \text{Fun}^\otimes(\mathcal{X}', \mathcal{X}).$$

The corresponding ∞ -category is denoted $\mathbf{Gpd}_\infty^\otimes$.

Since equivalence classes of left fibrations $\mathcal{X}^\otimes \rightarrow N\Gamma^{\text{op}}$ can be put into bijection with equivalence classes of functors $N\Gamma^{\text{op}} \rightarrow \mathbf{Kan}$, the definition above is essentially equivalent to the notion of *special Γ -space* we all learned in grade school [47].

11.2. **Proposition.** *The ∞ -category $\mathbf{Gpd}_\infty^\otimes$ is presentable.*

Proof. Consider the full simplicial subcategory $\mathcal{P}'(N\Gamma^{\text{op}})^{\Delta}$ of $s\mathbf{Set}_{/N\Gamma^{\text{op}}}^{\Delta}$ spanned by the left fibrations; its simplicial nerve $\mathcal{P}'(N\Gamma^{\text{op}})$ is presentable by [37, 5.1.1.1, 5.1.2.4, and 5.3.5.12]. For any pointed finite set I_+ , the forgetful functor $N\Gamma_{I_+}^{\text{op}} \rightarrow N\Gamma^{\text{op}}$ is a left fibration. One may therefore contemplate the set

$$S := \left\{ \coprod_{i \in I} N\Gamma_{\{i\}_+}^{\text{op}} \rightarrow N\Gamma_{I_+}^{\text{op}} \right\}_{I_+ \in N\Gamma^{\text{op}}}$$

of morphisms of $\mathcal{P}'(N\Gamma^{\text{op}})$. The ∞ -category $\mathbf{Gpd}_{\infty}^{\otimes}$ is the full subcategory of S -local objects; it is presentable by [37, 5.5.4.15]. \square

The preceding proof also exhibits the following.

11.2.1. Corollary. *The ∞ -category $\mathbf{Gpd}_{\infty}^{\otimes}$ is an accessible localization of $\mathcal{P}'(N\Gamma^{\text{op}})$.*

11.3. If \mathcal{X} is a symmetric monoidal ∞ -groupoid, then for any nonnegative integer $j \geq 0$, the canonical morphism

$$\pi_j \mathcal{X}_{\{1\}_+} \times \pi_j \mathcal{X}_{\{2\}_+} \cong \pi_j \mathcal{X}_{\{1,2\}_+} \rightarrow \pi_j \mathcal{X}_{\{\xi\}_+}$$

induced by the morphism $\mu: \{1,2\}_+ \rightarrow \{\xi\}_+$ that carries both 1 and 2 to ξ is a commutative monoid structure; we denote this monoid by $\pi_j \mathcal{X}$.

This is an apparent abuse of notation: $\pi_j \mathcal{X}$ does *not* refer to the homotopy groups of the space \mathcal{X} itself, but rather to the *set* $\pi_j \mathcal{X}_{\{\ast\}}$, equipped with the commutative monoid structure described above. However, this abuse is not serious: the Eckmann-Hilton argument, when $j \geq 1$, implies that this monoid structure coincides with the group structure on $\pi_j \mathcal{X}_{\ast}$. In particular, $\pi_j \mathcal{X}$ is an abelian group if $j \geq 1$.

11.4. Proposition. *The following are equivalent for a symmetric monoidal ∞ -groupoid \mathcal{X} .*

(11.4.1) *For any finite set I and any $i \in I$, the following condition obtains: the functor $\mathcal{X}_I \rightarrow \mathcal{X}_{(I-\{i\})}$ induced by the map $\alpha: I_+ \rightarrow (I-\{i\})_+$ given by*

$$\alpha(j) := \begin{cases} j & \text{if } j \neq i; \\ \ast & \text{if } j = i \end{cases}$$

and the functor $\mathcal{X}_I \rightarrow \mathcal{X}_{\{\xi\}}$ induced by the map $\beta: I_+ \rightarrow \{\xi\}_+$ such that $\beta(i) = \xi$ for every $i \in I$ together exhibit \mathcal{X}_I as a product of $\mathcal{X}_{(I-\{i\})}$ and $\mathcal{X}_{\{\xi\}}$.

(11.4.2) *For any nonempty finite set I , there exists an element $i \in I$ satisfying the condition above.*

(11.4.3) *As above, but only for $I = \{1,2\}$ and $i = 1$.*

(11.4.4) *The commutative monoid $\pi_0 \mathcal{X}$ is a group.*

Proof. Clearly each condition follows from the one preceding it. To see that the last condition implies the first, observe that since $\pi_j \mathcal{X}$ is a group for any $j \geq 0$, each homomorphism

$$\pi_j \mathcal{X}_I \rightarrow \pi_j \mathcal{X}_{\{\xi\}} \times \pi_j \mathcal{X}_{I-\{j\}}$$

is an isomorphism. \square

11.5. Definition. A *Picard ∞ -groupoid* is a symmetric monoidal ∞ -groupoid satisfying any of the equivalent conditions of Proposition 11.4. Denote by $\mathbf{Pic}_{\infty}^{\otimes}$ the full subcategory of $\mathbf{Gpd}_{\infty}^{\otimes}$ spanned by the Picard ∞ -groupoids.

11.6. Proposition. *The ∞ -category $\mathbf{Pic}_{\infty}^{\otimes}$ is an accessible localization of $\mathbf{Gpd}_{\infty}^{\otimes}$.*

Proof. Apply the left adjoint $L: \mathcal{P}'(N\Gamma^{\text{op}}) \rightarrow \mathbf{Gpd}_{\infty}^{\otimes}$ to the morphism

$$\phi: N\Gamma_{\{1\}_+}^{\text{op}} \sqcup N\Gamma_{\{\xi\}_+}^{\text{op}} \rightarrow N\Gamma_{\{1,2\}_+}^{\text{op}}$$

of $\mathcal{P}'(N\Gamma^{\text{op}})$; the ∞ -category $\mathbf{Pic}_{\infty}^{\otimes}$ is the full subcategory of $\mathbf{Gpd}_{\infty}^{\otimes}$ spanned by the $\{L\phi\}$ -local objects. The result now follows from [37, 5.5.4.15]. \square

11.7. Definition. Denote by $P: \mathbf{Gpd}_{\infty}^{\otimes} \rightarrow \mathbf{Pic}_{\infty}^{\otimes}$ the left adjoint described in the previous proposition. We shall call this functor *Picardification*.

11.8. Any Picard ∞ -groupoid \mathcal{X} may be turned into a spectrum by forming a left Kan extension to \mathbf{Kan}_* and then evaluating on spheres in order to construct a spectrum $\{\mathcal{X}(S^n)\}_{n \geq 0}$ [47]. It is a well-known consequence of Segal's delooping machine that this establishes an equivalence $\mathbf{Pic}_\infty^\otimes \xrightarrow{\sim} \mathbf{Sp}_{\geq 0}$ between the ∞ -category of Picard ∞ -groupoids and the ∞ -category of connective spectra [46].

We now aim to relate the Picardification to K -theory. More precisely, we shall now identify, for any symmetric monoidal ∞ -groupoid \mathcal{X} , the K -theory of the “total space” of \mathcal{X} with the Picardification of a related symmetric monoidal ∞ -groupoid given by a *wreath product*.

11.9. We may endow the total space \mathcal{X}^\otimes of any inner fibration $\mathcal{X}^\otimes \rightarrow N\Gamma^{\text{op}}$ with the structure of a pair of ∞ -categories simply by declaring $\mathcal{X}_\dagger^\otimes := \mathcal{X} \times_{N\Gamma^{\text{op}}} N\Gamma_\dagger^{\text{op}}$.

11.10. **Proposition.** *If $p: \mathcal{X}^\otimes \rightarrow N\Gamma^{\text{op}}$ is a symmetric monoidal ∞ -groupoid, then $(\mathcal{X}^\otimes, \mathcal{X}_\dagger^\otimes)$ is a Waldhausen ∞ -category.*

Proof. Any point of the fiber $\mathcal{X}_\emptyset^\otimes$ is a zero object of \mathcal{X}^\otimes , and any edge that covers the inclusion $* \rightarrow I_+$ is by definition ingressive. The natural square

$$\begin{array}{ccc} \mathcal{O}_\dagger(\mathcal{X}^\otimes) & \longrightarrow & \mathcal{O}_\dagger(N\Gamma^{\text{op}}) \\ s \downarrow & & \downarrow s \\ \mathcal{X}^\otimes & \longrightarrow & N\Gamma^{\text{op}} \end{array}$$

is a pullback, so $s: \mathcal{O}_\dagger(\mathcal{X}^\otimes) \rightarrow \mathcal{X}^\otimes$ is a cocartesian fibration, and for any s -cocartesian edge η , the edge $p(t(\eta))$ is ingressive, so $t(\eta)$ is also ingressive. \square

11.11. The assignment $(\mathcal{X}^\otimes/N\Gamma^{\text{op}}) \mapsto (\mathcal{X}^\otimes, \mathcal{X}_\dagger^\otimes)$ defines a functor $T: \mathbf{Gpd}_\infty^\otimes \rightarrow \mathbf{Wald}_\infty$.

The Waldhausen ∞ -categories that are attached to a symmetric monoidal ∞ -groupoid are of a very special kind: the cofibrations therein split in a canonical fashion. To make this precise, let us introduce an enlargement of our category \mathbf{M} .

11.12. **Notation.** Write $\tilde{\mathbf{M}}$ for the ordinary category whose objects are pairs of integers (m, i) , where $m \geq 0$ and $0 \leq i \leq m$. A map $(n, j) \rightarrow (m, i)$ is a morphism $\phi: [\mathbf{m}] \rightarrow [\mathbf{n}]$ such that either $j \leq \phi(i)$ or both $i = m$ and $j = n$. We have an obvious inclusion $\mathbf{M} \hookrightarrow \tilde{\mathbf{M}}$. Again we can endow $N\tilde{\mathbf{M}}$ with the structure of a pair by setting $N\tilde{\mathbf{M}}_\dagger := N\mathbf{M}_\dagger$.

For any Waldhausen ∞ -category \mathcal{C} , write $\mathcal{G}\mathcal{C}$ for the simplicial set over $N\Delta^{\text{op}}$ defined by the following universal property. We require, for any simplicial set K and any map $\sigma: K \rightarrow N\Delta^{\text{op}}$, a bijection

$$\text{Mor}_{/N\Delta^{\text{op}}}(K, \mathcal{G}(\mathcal{C})) \cong \text{Mor}_{s\mathbf{Set}(2)}((K \times_{N\Delta^{\text{op}}} N\tilde{\mathbf{M}}, K \times_{N\Delta^{\text{op}}} (N\tilde{\mathbf{M}})_\dagger), (\mathcal{C}, \mathcal{C}_\dagger)),$$

functorial in σ . We let $\mathcal{T}\mathcal{C} \subset \mathcal{G}\mathcal{C}$ be the full simplicial subset spanned by the totally filtered objects. The inclusion $\mathbf{M} \hookrightarrow \tilde{\mathbf{M}}$ induces inner fibrations $\mathcal{G}\mathcal{C} \rightarrow \mathcal{T}\mathcal{C}$ and $\mathcal{T}\mathcal{C} \rightarrow \mathcal{S}\mathcal{C}$; we abuse notation slightly by writing $\iota\mathcal{T}\mathcal{C}$ for the fiber product $\iota\mathcal{S}\mathcal{C} \times_{\mathcal{S}\mathcal{C}} \mathcal{T}\mathcal{C}$.

11.13. **Definition.** Suppose \mathcal{C} a Waldhausen ∞ -category; we say that *cofibrations in \mathcal{C} split* if the map $\iota\mathcal{T}\mathcal{C} \rightarrow \iota\mathcal{S}\mathcal{C}$ is a weak homotopy equivalence.

11.14. **Proposition.** *Cofibrations in $N\Gamma^{\text{op}}$ split.*

Proof. For any totally filtered objects $J_+: (\Delta^n)^\sharp \rightarrow N\Gamma^{\text{op}}$ and $I_+: (\Delta^m)^\sharp \rightarrow N\Gamma^{\text{op}}$ and for any cocartesian edge $\eta: J_+ \rightarrow I_+$ of $\mathcal{S}(N\Gamma^{\text{op}})$ covering $\phi: [\mathbf{n}] \rightarrow [\mathbf{m}]$ in $N\Delta^{\text{op}}$, we may construct a map $J_{n,+} \rightarrow I_{m,+}$ as the composite

$$J_{n,+} \xrightarrow{r} J_{\phi(m),+} \xrightarrow{\eta} I_{m,+},$$

where $r: J_{n,+} \rightarrow J_{\phi(m),+}$ is the unique retraction of $J_{\phi(m),+} \rightarrowtail J_{n,+}$ whose pullback

$$J_{n,+} \times_{J_{\phi(m),+}} J_{\phi(m)} \rightarrow J_{\phi(m)}$$

is a bijection. It is now easy to see at the level of ordinary categories that this defines a homotopy inverse to the map $\iota\mathcal{T}(N\Gamma^{\text{op}}) \rightarrow \iota\mathcal{S}(N\Gamma^{\text{op}})$. \square

11.14.1. **Corollary.** *For any symmetric monoidal ∞ -groupoid \mathcal{X} , cofibrations in \mathcal{X}^\otimes split.*

Now we can introduce the wreath product.

11.15. **Notation.** Denote by $\Gamma^{\text{op}} \wr \Gamma^{\text{op}}$ the following ordinary category. The objects are pairs (I, J_I) , where I is a finite set and $J_I = (J_i)_{i \in I}$ is an I -tuple of finite sets. A morphism $(K, L_K) \rightarrow (I, J_I)$ is a map $\phi: K_+ \rightarrow I_+$ of pointed finite sets and an I -tuple $(\psi_k: L_{k,+} \rightarrow J_{\phi(k),+})_{k \in K}$ of maps of pointed finite sets. The functor

$$(I, J_I) \mapsto \bigvee_{i \in I} J_{i,+}$$

defines a functor $u': \Gamma^{\text{op}} \wr \Gamma^{\text{op}} \rightarrow \Gamma^{\text{op}}$. The category $\Gamma^{\text{op}} \wr \Gamma^{\text{op}}$ is described as the Leinster category of $F \wr F$ in [2], as the category \mathcal{A} in [42], and (after passing to nerves) as the wreath product of the terminal ∞ -operad with itself in [41].

It is easy to see that the functor $\Gamma^{\text{op}} \wr \Gamma^{\text{op}} \rightarrow \Gamma^{\text{op}}$ given by the assignment $(I, J_I) \mapsto I$ is both a Grothendieck fibration and a Grothendieck opfibration. We will regard the nerve $N(\Gamma^{\text{op}} \wr \Gamma^{\text{op}}) \rightarrow N\Gamma^{\text{op}}$ as a cocartesian fibration, and we will write $Z(N\Gamma^{\text{op}}) := \iota_{N\Gamma^{\text{op}}} N(\Gamma^{\text{op}} \wr \Gamma^{\text{op}})$; in other words, $Z(N\Gamma^{\text{op}})$ is the subcategory of $N(\Gamma^{\text{op}} \wr \Gamma^{\text{op}})$ containing all objects and those morphisms $(K, L_K) \rightarrow (I, J_I)$ such that for any $i \in I$, the maps $L_{k,+} \rightarrow J_{i,+}$ with $k \in \phi^{-1}(i)$ exhibit $J_{i,+}$ as the coproduct $\bigvee_{k \in \phi^{-1}(i)} L_{k,+}$. The ∞ -category $Z(N\Gamma^{\text{op}})$ comes equipped with a left fibration $p: Z(N\Gamma^{\text{op}}) \rightarrow N\Gamma^{\text{op}}$ and a functor $u := N(u')|_{Z(N\Gamma^{\text{op}})}: Z(N\Gamma^{\text{op}}) \rightarrow N\Gamma^{\text{op}}$.

11.16. **Definition.** The *wreath product* $Z(\mathcal{X})$ of a symmetric monoidal ∞ -groupoid $\mathcal{X}^\otimes \rightarrow N\Gamma^{\text{op}}$ with $Z(N\Gamma^{\text{op}})$ is the fiber product $Z(N\Gamma^{\text{op}}) \times_{u, N\Gamma^{\text{op}}} \mathcal{X}^\otimes$. It is equipped with the composite left fibration

$$Z(\mathcal{X}) \rightarrow Z(N\Gamma^{\text{op}}) \xrightarrow{p} N\Gamma^{\text{op}}.$$

It is a straightforward matter to show that the wreath product defines a functor

$$Z: \mathbf{Gpd}_\infty^\otimes \rightarrow \mathbf{Gpd}_\infty^\otimes.$$

This wreath product is of course a special case of a far more general construction, but we will not need the added generality here. Our only objective here is the following.

11.17. **Proposition.** *The composite $\mathbf{K}^{\text{conn}} \circ T$ is naturally equivalent to the composite $P \circ Z$, where we identify $\mathbf{Pic}_\infty^\otimes$ with the ∞ -category $\mathbf{Sp}_{\geq 0}$ of connective spectra by 11.8.*

Proof. Consider the functor $v_+: N\Delta^{\text{op}} \rightarrow N\Gamma^{\text{op}}$ given by the formula

$$v(E)_+ := \text{Mor}_\Delta(E, [1]) / \{0, 1\},$$

where $0, 1 \in \text{Mor}_\Delta(E, [1])$ are the constant maps $E \rightarrow [1]$. Clearly any element α of the unpointed set $v(E)$ is uniquely specified by α_0 , the largest element of E that is carried to 0 under α , or by α_1 , the smallest element of E that is carried to 1 under α .

Working in the category of ordinary categories, we form a homotopy pullback square

$$\begin{array}{ccc} \iota\mathcal{T}(N\Gamma^{\text{op}}) & \xrightarrow{g} & Z(N\Gamma^{\text{op}}) \\ \downarrow & & \downarrow p \\ N\Delta^{\text{op}} & \xrightarrow{v_+} & N\Gamma^{\text{op}}, \end{array}$$

where the functor $g: \iota\mathcal{T}(N\Gamma^{\text{op}}) \rightarrow Z(N\Gamma^{\text{op}})$ carries an object (J, I) — where $J \in \Delta^{\text{op}}$ and $I: (\Delta^J)^\sharp \rightarrow N\Gamma^{\text{op}}$ is a totally filtered object — to the object

$$(v(J)_+, ((I_{\alpha_1} - I_{\alpha_0})_+)_{\alpha \in u(E)}) \in Z(N\Gamma^{\text{op}}).$$

Note that the composite $u \circ g: \iota\mathcal{T}(N\Gamma^{\text{op}}) \rightarrow N\Gamma^{\text{op}}$ is the functor induced by the section $N\Delta^{\text{op}} \rightarrow N\widetilde{M}$ given by $[\mathbf{m}] \mapsto (m, m)$.

We now give a functor $\psi: \iota\mathcal{T}\mathcal{X} \rightarrow Z(\mathcal{X})$, functorial in the symmetric monoidal ∞ -groupoid \mathcal{X} , such that every square of the resulting diagram

$$\begin{array}{ccccc} \iota\mathcal{T}\mathcal{X} & \xrightarrow{\psi} & Z(\mathcal{X}) & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow & & \downarrow \\ \iota\mathcal{T}(N\Gamma^{\text{op}}) & \xrightarrow{v} & Z(N\Gamma^{\text{op}}) & \xrightarrow{u} & N\Gamma^{\text{op}} \\ \downarrow & & \downarrow p & & \\ N\Delta^{\text{op}} & \longrightarrow & N\Gamma^{\text{op}} & & \end{array}$$

is a homotopy pullback. Since cofibrations split in \mathcal{X} , the functor $\iota\mathcal{T}\mathcal{X} \rightarrow \iota\mathcal{S}\mathcal{X}$ is an equivalence, and the result will then follow immediately from the universal properties of K and P .

To define ψ , it is enough to observe that the section $N\Delta^{\text{op}} \rightarrow N\tilde{M}$ given by $[\mathbf{m}] \mapsto (m, m)$ defines a functor $\iota\mathcal{T}\mathcal{X} \rightarrow \mathcal{X}$ (functorial in \mathcal{X}) for which the diagram

$$\begin{array}{ccc} \iota\mathcal{T}\mathcal{X} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \iota\mathcal{T}(N\Gamma^{\text{op}}) & \xrightarrow{u \circ g} & N\Gamma^{\text{op}} \end{array}$$

is a homotopy pullback. □

12. EXAMPLE: WALDHAUSEN A -THEORY OF HIGHER TOPOI

From any ∞ -topos we may extract a Waldhausen ∞ -category. This permits us to speak of Waldhausen's A -theory of these ∞ -topoi. Here, we construct this theory as a generalization of Waldhausen's A -theory, and we show that this theory enjoys some pleasant formal properties.

We emphasize that we only give the very beginnings of the theory here, as an illustration of the techniques developed in this paper. A complete treatment would involve variants of THH that would be well adapted to this context. Such theories lie outside our current scope, but we hope to develop these in future work. We would also like to remark that some of the results that follow were also discovered independently by Rune Haugseng.

We begin with a construction of A -theory that is valid in any ∞ -topos.

12.1. Lemma. *Suppose \mathcal{E} an ∞ -topos. Then the functor*

$$\text{Fun}(\Delta^2, \mathcal{E}) \rightarrow \text{Fun}(\Delta^{\{0,2\}}, \mathcal{E})$$

is both a cartesian fibration and a cocartesian fibration.

Proof. This follows directly from the existence of pullbacks and pushouts in \mathcal{E} . □

12.2. Notation. Suppose \mathcal{E} an ∞ -topos. We consider the fibration

$$\text{Fun}(\Delta^2/\Delta^{\{0,2\}}, \mathcal{E}) \cong \text{Fun}(\Delta^2, \mathcal{E}) \times_{\text{Fun}(\Delta^{\{0,2\}}, \mathcal{E})} \mathcal{E} \rightarrow \mathcal{E},$$

which is both cartesian and cocartesian. Write $\mathcal{R}(\mathcal{E}) \subset \text{Fun}(\Delta^2/\Delta^{\{0,2\}}, \mathcal{E})$ for the full subcategory spanned by those retract diagrams $X \rightarrow Y \rightarrow X$ that are compact as objects of the fiber $\text{Fun}(\Delta^2/\Delta^{\{0,2\}}, \mathcal{E}) \times_{\mathcal{E}} \{X\}$. Write $\mathcal{R}_+(\mathcal{E})$ for the fiber product $\text{Fun}(\Delta^2/\Delta^{\{0,2\}}, \mathcal{E}) \times_{\mathcal{E}} \iota\mathcal{E}$. For any object X of \mathcal{E} , write $\mathcal{R}(\mathcal{E})_X$ for the fiber of the functor $\mathcal{R}(\mathcal{E}) \rightarrow \mathcal{E}$.

12.3. Proposition. *Suppose \mathcal{E} an ∞ -topos. Then the functor $\mathcal{R}(\mathcal{E}) \rightarrow \mathcal{E}$ is a Waldhausen cocartesian fibration.*

Proof. For any object X of \mathcal{E} , the ∞ -category $\mathcal{R}(\mathcal{E})_X$ clearly has a zero object. Since finite colimits of compact objects are compact, the ∞ -category $\mathcal{R}(\mathcal{E})_X$ with its maximal pair structure is a Waldhausen ∞ -category. Now note that for any edge $g: X \rightarrow Y$, the functors

$$\text{Fun}(\Delta^2, \mathcal{E}) \times_{\text{Fun}(\Delta^{\{0,2\}}, \mathcal{E})} \{\text{id}_Y\} \rightarrow \text{Fun}(\Delta^2, \mathcal{E}) \times_{\text{Fun}(\Delta^{\{0,2\}}, \mathcal{E})} \{g\} \rightarrow \text{Fun}(\Delta^2, \mathcal{E}) \times_{\text{Fun}(\Delta^{\{0,2\}}, \mathcal{E})} \{\text{id}_X\}$$

each preserve filtered colimits; indeed, the first functor preserves all colimits, since colimits are universal in \mathcal{E} , and the second preserves filtered colimits by a simple argument. Since the functor

$$g_! : \mathrm{Fun}(\Delta^2, \mathcal{E}) \times_{\mathrm{Fun}(\Delta^{\{0,2\}}, \mathcal{E})} \{\mathrm{id}_X\} \longrightarrow \mathrm{Fun}(\Delta^2, \mathcal{E}) \times_{\mathrm{Fun}(\Delta^{\{0,2\}}, \mathcal{E})} \{\mathrm{id}_Y\}$$

induced by g is left adjoint to the composite above, it preserves compact objects. Hence it restricts to a functor $g_! : \mathcal{R}(\mathcal{E})_X \longrightarrow \mathcal{R}(\mathcal{E})_Y$, which preserves finite colimits. \square

12.4. Proposition. *For any ∞ -topos \mathcal{E} , the functor $R(\mathcal{E}) : \mathcal{E} \longrightarrow \mathbf{Wald}_\infty$ classified by the Waldhausen cocartesian fibration $\mathcal{R}(\mathcal{E}) \longrightarrow \mathcal{E}$ preserves filtered colimits.*

Proof. We note that $R(\mathcal{E})(X)$ can be identified with the compact objects of the ∞ -category $\mathcal{E}_{X/}/_X$. One may thus express $R(\mathcal{E})$ as the composite of the functor $\widehat{\mathcal{R}}(\mathcal{E}) : \mathcal{E} \longrightarrow \mathbf{Pr}_\omega^L$ that classifies the fibration $\mathrm{Fun}(\Delta^2/\Delta^{\{0,2\}}, \mathcal{E}) \longrightarrow \mathcal{E}$ with the functor θ from [37, Pr. 5.5.7.8] (with $\kappa = \omega$). Note that the functor $\widehat{\mathcal{R}}(\mathcal{E})$ preserves filtered colimits. Thanks to [37, Pr. 5.5.7.8], the functor $R(\mathcal{E})$, regarded as taking values in the full subcategory of $\mathbf{Cat}_\infty(\kappa_1)^{\mathrm{Rex}}$ spanned by the essentially small and idempotent complete ∞ -categories, also preserves filtered colimits. Now [37, Pr. 5.5.7.11], combined with the fact that a filtered colimit of idempotent complete ∞ -categories is idempotent complete, guarantees that $R(\mathcal{E})$ preserves filtered colimits when regarded as taking values in \mathbf{Cat}_∞ . Now Pr. 4.4 applies to yield the result. \square

This result permits us to study $\mathcal{R}(\mathcal{E})$ by studying its pullback $\mathcal{R}(\mathcal{E}) \times_{\mathcal{E}} \mathcal{E}^\omega$ to the full subcategory \mathcal{E}^ω spanned by the compact objects of \mathcal{E} . There, it turns out, we find more structure, in the form of transfer maps.

12.5. Definition. We say that a morphism $g : X \longrightarrow Y$ of an ∞ -topos \mathcal{E} is *small* if the pullback functor $g^* : \mathcal{E}_{/Y} \longrightarrow \mathcal{E}_{/X}$ preserves compact objects, or equivalently [37, Pr. 5.5.7.2], if its right adjoint $g_* : \mathcal{E}_{/X} \longrightarrow \mathcal{E}_{/Y}$ preserves filtered colimits. Write \mathcal{E}_ω for the subcategory containing all the objects whose morphisms are small.

12.6. Proposition. *Suppose \mathcal{E} an ∞ -topos. Then the functor*

$$\mathcal{R}(\mathcal{E}) \times_{\mathcal{E}} \mathcal{E}_\omega \longrightarrow \mathcal{E}_\omega$$

is a Waldhausen cartesian fibration as well as a Waldhausen cocartesian fibration.

Proof. This follows immediately from the fact that colimits are universal in \mathcal{E} . \square

Since colimits in an ∞ -topos \mathcal{E} are universal, an easy argument guarantees that any morphism between compact objects of \mathcal{E} is small. Hence we have the following.

12.6.1. Corollary. *Suppose \mathcal{E} an ∞ -topos. Then the functor*

$$\mathcal{R}(\mathcal{E}) \times_{\mathcal{E}} \mathcal{E}^\omega \longrightarrow \mathcal{E}^\omega$$

is a Waldhausen cartesian fibration as well as a Waldhausen cocartesian fibration.

12.7. Proposition. *Suppose \mathcal{E} an ∞ -topos. Then the functor $\mathcal{E}^{\omega, \mathrm{op}} \longrightarrow \mathbf{Wald}_\infty$ classified by the Waldhausen cartesian fibration $\mathcal{R}(\mathcal{E}) \times_{\mathcal{E}} \mathcal{E}^\omega \longrightarrow \mathcal{E}^\omega$ preserves all finite limits.*

Proof. This follows immediately from the fact that the class of all morphisms in \mathcal{E} is local [37, Pr. 6.1.3.10]. \square

12.8. Notation. For any morphism $g : X \longrightarrow Y$ of an ∞ -topos \mathcal{E} , write $g_! : \mathcal{R}(\mathcal{E})_X \longrightarrow \mathcal{R}(\mathcal{E})_Y$ for the exact functor arising from the Waldhausen cocartesian fibration $\mathcal{R}(\mathcal{E}) \times_{\mathcal{E}, g} \Delta^1 \longrightarrow \Delta^1$. If g is small, write $g^* : \mathcal{R}(\mathcal{E})_Y \longrightarrow \mathcal{R}(\mathcal{E})_X$ for the exact functor corresponding to the Waldhausen cartesian fibration

$$\mathcal{R}(\mathcal{E}) \times_{\mathcal{E}, g} \Delta^1 \longrightarrow \Delta^1.$$

Another immediate consequence of the universality of colimits in an ∞ -topos \mathcal{E} is the following base change theorem.

12.9. Proposition. *Suppose \mathcal{E} an ∞ -topos. For any pullback square*

$$\begin{array}{ccc} X' & \xrightarrow{\psi} & X \\ g \downarrow & & \downarrow g \\ Y' & \xrightarrow{\psi} & Y \end{array}$$

in \mathcal{E} in which the morphisms abusively labeled ψ are small, the induced square

$$\begin{array}{ccc} \mathcal{R}(\mathcal{E})_{X'} & \xrightarrow{\psi_!} & \mathcal{R}(\mathcal{E})_X \\ g_! \downarrow & & \downarrow g_! \\ \mathcal{R}(\mathcal{E})_{Y'} & \xrightarrow{\psi_!} & \mathcal{R}(\mathcal{E})_Y \end{array}$$

is right adjointable [37, Df. 7.3.1.2]; that is, the natural morphism

$$g_! \circ \psi^* \longrightarrow \psi^* \circ g_!$$

of $\mathrm{Fun}_{\mathbf{Wald}_\infty}^b(\mathcal{R}(\mathcal{E})_X, \mathcal{R}(\mathcal{E})_{Y'})$ is an equivalence.

12.10. Definition. For any ∞ -topos \mathcal{E} , the *Waldhausen A -theory of \mathcal{E}* , $A_\mathcal{E}: \mathcal{E} \rightarrow \mathbf{Kan}_*$, is the composite $K \circ R(\mathcal{E})$, where the functor $R(\mathcal{E}): \mathcal{E} \rightarrow \mathbf{Wald}_\infty$ classifies the Waldhausen cocartesian fibration $\mathcal{R}(\mathcal{E}) \rightarrow \mathcal{E}$. Similarly, the functor $\mathbf{A}_\mathcal{E}$ is defined as $\mathbf{K}^{\mathrm{conn}} \circ R(\mathcal{E})$; of course $A_\mathcal{E} \simeq \Omega^\infty \circ \mathbf{A}_\mathcal{E}$.

Similarly, the *transfer Waldhausen A -theory of \mathcal{E}* , $A_\mathcal{E}^{\mathrm{tr}}: \mathcal{E}_\omega^{\mathrm{op}} \rightarrow \mathbf{Kan}_*$, is the composite $K \circ R^{\mathrm{tr}}(\mathcal{E})$, where the functor $R^{\mathrm{tr}}(\mathcal{E}): \mathcal{E}_\omega^{\mathrm{op}} \rightarrow \mathbf{Wald}_\infty$ classifies the Waldhausen cartesian fibration $\mathcal{R}(\mathcal{E}) \times_\mathcal{E} \mathcal{E}_\omega \rightarrow \mathcal{E}_\omega$. Similarly, the functor $\mathbf{A}_\mathcal{E}^{\mathrm{tr}}$ is defined as $\mathbf{K}^{\mathrm{conn}} \circ R^{\mathrm{tr}}(\mathcal{E})$; of course $A_\mathcal{E}^{\mathrm{tr}} \simeq \Omega^\infty \circ \mathbf{A}_\mathcal{E}^{\mathrm{tr}}$.

12.11. Construction. The preceding definition ensures that $A_\mathcal{E}$ is well-defined up to a contractible ambiguity. To obtain an explicit model of $A_\mathcal{E}$, we proceed in the following manner. Apply \mathcal{S} to the Waldhausen cocartesian fibration $\mathcal{R}(\mathcal{E}) \rightarrow \mathcal{E}$ to obtain a Waldhausen cocartesian fibration $\mathcal{S}\mathcal{R}(\mathcal{E}) \rightarrow N\Delta^{\mathrm{op}} \times \mathcal{E}$, and consider the subcategory $\iota_{N\Delta^{\mathrm{op}} \times \mathcal{E}} \mathcal{S}\mathcal{R}(\mathcal{E}) \subset \mathcal{S}\mathcal{R}(\mathcal{E})$ consisting of cocartesian edges. The composite

$$\iota_{N\Delta^{\mathrm{op}} \times \mathcal{E}} \mathcal{S}\mathcal{R}(\mathcal{E}) \longrightarrow N\Delta^{\mathrm{op}} \times \mathcal{E} \longrightarrow \mathcal{E}$$

is now a left fibration with a contractible space of sections given by

$$\mathcal{E} \cong \{0\} \times \mathcal{E} \xleftarrow{\sim} \iota_{\mathcal{S}0} \mathcal{R}(\mathcal{E}) \hookrightarrow \iota_{N\Delta^{\mathrm{op}} \times \mathcal{E}} \mathcal{S}\mathcal{R}(\mathcal{E}).$$

It is clear by construction that this left fibration classifies a functor $B_\mathcal{E}: \mathcal{E} \rightarrow \mathbf{Kan}_*$ such that $A_\mathcal{E} \simeq \Omega \circ B_\mathcal{E}$.

A similar construction can be applied to the coWaldhausen cocartesian fibration

$$(\mathcal{R}(\mathcal{E}) \times_\mathcal{E} \mathcal{E}_\omega)^{\mathrm{op}} \rightarrow \mathcal{E}_\omega^{\mathrm{op}}$$

yields (a singly delooped copy of) the functor $A_\mathcal{E}^{\mathrm{tr}}$.

Note that Pr. 12.9 guarantees the A -theory functors $\mathbf{A}_\mathcal{E}$ and $\mathbf{A}_\mathcal{E}^{\mathrm{tr}}$ together satisfy a base change compatibility.

12.12. Proposition. *For any pullback square*

$$\begin{array}{ccc} X' & \xrightarrow{\psi} & X \\ g \downarrow & & \downarrow g \\ Y' & \xrightarrow{\psi} & Y \end{array}$$

of any ∞ -topos \mathcal{E} in which the morphisms abusively labeled ψ are small, the induced square

$$\begin{array}{ccc} \mathbf{A}_\mathcal{E}(X) & \xrightarrow{\psi^*} & \mathbf{A}_\mathcal{E}(X') \\ g_! \downarrow & & \downarrow g_! \\ \mathbf{A}_\mathcal{E}(Y) & \xrightarrow{\psi^*} & \mathbf{A}_\mathcal{E}(Y') \end{array}$$

commutes.

The following is an immediate consequence of Cor. 8.2.1 and Pr. 8.3.

12.13. Proposition. *For any object X of any ∞ -topos \mathcal{E} , there is a natural equivalence*

$$\mathbf{A}_{\mathcal{E}}(X) \xrightarrow{\sim} \mathbf{K}^{\text{conn}}(\mathbf{Sp}(\mathcal{E}_{X/}/X)^{\omega}).$$

12.14. When $\mathcal{E} = \mathbf{Kan}$, Cor. 10.8.3 guarantees that the Waldhausen A -theory functor agrees with K -theory of *finitely dominated retractive spaces* considered in [21, §6]. It differs from A -theory as defined in [57, §2], but only on π_0 , since our $\mathcal{R}(\mathcal{E})_X$ is the idempotent completion of the relative nerve of Waldhausen's category with cofibrations and weak equivalences.

12.15. Construction. Let us remark also that the construction of the Waldhausen A -theory functor $A_{\mathcal{E}}$ given here includes the data of a collection of assembly maps. Indeed, for any compactly generated ∞ -topos \mathcal{E} , functoriality alone provides maps

$$\text{Map}_{\mathcal{E}}(X, Y) \longrightarrow \text{Map}_{\mathbf{Sp}_{\geq 0}}(\mathbf{A}_{\mathcal{E}}(X), \mathbf{A}_{\mathcal{E}}(Y))$$

and thus maps

$$\mathbf{A}_{\mathcal{E}}(X) \wedge \text{Map}_{\mathcal{E}}(X, Y)_+ \longrightarrow \mathbf{A}_{\mathcal{E}}(Y).$$

If X is the terminal object $*$ of \mathcal{E} , then we obtain more familiar-looking assembly maps:

$$\mathbf{A}_{\mathcal{E}}(*) \wedge \Gamma_{\mathcal{E},+}(Y) \longrightarrow \mathbf{A}_{\mathcal{E}}(Y),$$

where $\Gamma_{\mathcal{E},+}$ is the “global sections functor” with a disjoint basepoint, $\text{Map}_{\mathcal{E}}(*, -)_+$. These are functorial in Y , in the sense that we obtain an *assembly morphism*

$$\alpha: \mathbf{A}_{\mathcal{E}}(*) \wedge \Gamma_{\mathcal{E},+} \longrightarrow \mathbf{A}_{\mathcal{E}},$$

of $\text{Fun}(\mathcal{E}, \mathbf{Sp}_{\geq 0})$.

Furthermore, we apply K -theory to the obvious inclusion $N\Gamma^{\text{op}} \hookrightarrow \mathcal{R}(\mathcal{E})_*$ to obtain a map $\eta: \mathbf{S}^0 \longrightarrow \mathbf{A}_{\mathcal{E}}(*)$; call this the *unit morphism*. Composing η with the assembly map α , we obtain a natural transformation

$$e: \Sigma_+^{\infty} \circ \Gamma_{\mathcal{E}} \longrightarrow \mathbf{A}_{\mathcal{E}}.$$

More explicitly, for any object X of \mathcal{E} , we have a map

$$\Gamma_{\mathcal{E}}(X) \longrightarrow \text{Map}_{\mathbf{Wald}_{\infty}}(N\Gamma^{\text{op}}, \mathcal{R}(\mathcal{E})_X)$$

that carries a global section $x \in \Gamma_{\mathcal{E}}(X)$ to the unique exact functor $N\Gamma^{\text{op}} \longrightarrow \mathcal{R}(\mathcal{E})_X$ that carries any pointed finite set T to $T \vee (X, x)$ as an object over and under X . After applying the K -theory functor, we obtain a map

$$\Gamma_{\mathcal{E}}(X) \longrightarrow \Omega^{\infty} \mathbf{A}_{\mathcal{E}}(X) \simeq A_{\mathcal{E}}(X),$$

whence we obtain the component e_X .

We now give a construction that will display the functoriality of $\mathbf{A}_{\mathcal{E}}$ in the ∞ -topos \mathcal{E} . Since $\mathcal{R}(\mathcal{E})_X$ is canonically equivalent to $\mathcal{R}(\mathcal{E}/X)_X$, we can express functoriality in objects of \mathcal{E} as a special case of functoriality in geometric morphisms of ∞ -topoi. Consequently, we restrict our attention to the A -theory of terminal objects of ∞ -topoi.

12.16. Construction. Write ${}^L\mathbf{Top}_{\infty}$ for the ∞ -category of ∞ -topoi and geometric morphisms π^* . We may now pull back the universal cartesian fibration $p: \mathcal{Z} \longrightarrow \mathbf{Cat}_{\infty}(\kappa_1)^{\text{op}}$ of [37, §3.3.2] along the forgetful functor ${}^L\mathbf{Top}_{\infty} \longrightarrow \mathbf{Cat}_{\infty}(\kappa_1)$ to obtain a cartesian fibration

$$p: \mathcal{Z}_{\mathbf{Top}} := \mathcal{Z} \times_{\mathbf{Cat}_{\infty}(\kappa_1)^{\text{op}}} {}^L\mathbf{Top}_{\infty}^{\text{op}} \longrightarrow {}^L\mathbf{Top}_{\infty}^{\text{op}}.$$

An object of $\mathcal{Z}_{\mathbf{Top}}$ can be described as a pair (\mathcal{E}, Z) consisting of an ∞ -topos and an object Z thereof, and a morphism $(\mathcal{E}', Z') \longrightarrow (\mathcal{E}, Z)$ can be described as a pair (π^*, η) consisting of a geometric morphism of ∞ -topoi $\pi^*: \mathcal{E} \longrightarrow \mathcal{E}'$ and a morphism $\eta: Z' \longrightarrow \pi^*Z$. The functor given informally by $\mathcal{E} \longmapsto (\mathcal{E}, *_\mathcal{E})$ gives a section $s: {}^L\mathbf{Top}_{\infty}^{\text{op}} \longrightarrow \mathcal{Z}_{\mathbf{Top}}$ of p .

Note that over any edge $\mathcal{E}' \longrightarrow \mathcal{E}$ of ${}^L\mathbf{Top}_{\infty}^{\text{op}}$ that corresponds to an étale morphism of ∞ -topoi and for any object X' of \mathcal{E}' , one also has a cocartesian edge $(\mathcal{E}', X') \longrightarrow (\mathcal{E}, \pi_! X')$. Consequently, the pullback of p

$$\mathcal{Z}_{\mathbf{Top}} \times_{{}^L\mathbf{Top}_{\infty}^{\text{op}}} {}^L\mathbf{Top}_{\infty, \text{ét}}^{\text{op}} \longrightarrow {}^L\mathbf{Top}_{\infty, \text{ét}}^{\text{op}}$$

to the subcategory ${}^L\mathbf{Top}_{\infty, \acute{e}t}^{\text{op}} \subset {}^L\mathbf{Top}_{\infty}^{\text{op}}$ consisting of the étale morphisms of ∞ -topoi is both a cartesian fibration and a cocartesian fibration.

Now define a simplicial set \mathcal{Q} over ${}^L\mathbf{Top}_{\infty}^{\text{op}}$ via the following universal property. We require, for any simplicial set K and any map $\sigma: K \rightarrow {}^L\mathbf{Top}_{\infty}^{\text{op}}$, a bijection

$$\text{Mor}_{/{}^L\mathbf{Top}_{\infty}^{\text{op}}}(K, \mathcal{Q}) \cong \text{Mor}_{/{}^L\mathbf{Top}_{\infty}^{\text{op}}}(K \times \Delta^1, \mathcal{Z}_{\mathbf{Top}})$$

functorial in σ . The functor $\mathcal{Q} \rightarrow {}^L\mathbf{Top}_{\infty}^{\text{op}}$ is a cartesian fibration, and its pullback

$$\mathcal{Q} \times_{{}^L\mathbf{Top}_{\infty}^{\text{op}}} {}^L\mathbf{Top}_{\infty, \acute{e}t}^{\text{op}} \rightarrow {}^L\mathbf{Top}_{\infty, \acute{e}t}^{\text{op}}$$

is both a cartesian fibration and a cocartesian fibration thanks to [37, Cor. 3.2.2.13]. The inclusion $\Delta^{\{0\}} \hookrightarrow \Delta^1$ induces a functor $q: \mathcal{Q} \rightarrow \mathcal{Z}_{\mathbf{Top}}$ over ${}^L\mathbf{Top}_{\infty}^{\text{op}}$, which is both a cartesian fibration and a cocartesian fibration. Now we pull back q along the section s described above to obtain a functor

$$\mathcal{Q} \times_{\mathcal{Z}_{\mathbf{Top}}} {}^L\mathbf{Top}_{\infty}^{\text{op}} \rightarrow {}^L\mathbf{Top}_{\infty}^{\text{op}}$$

that is both a cartesian fibration and a cocartesian fibration.

An object of the fiber product $\mathcal{Q} \times_{\mathcal{Z}_{\mathbf{Top}}} {}^L\mathbf{Top}_{\infty}^{\text{op}}$ can be described as a pair (\mathcal{E}, X) consisting of an ∞ -topos \mathcal{E} and a pointed object X of \mathcal{E} , and a morphism $(\mathcal{E}', X') \rightarrow (\mathcal{E}, X)$ of $\mathcal{Q} \times_{\mathcal{Z}_{\mathbf{Top}}} {}^L\mathbf{Top}_{\infty}^{\text{op}}$ can be described as a pair (π^*, η) , where $\pi^*: \mathcal{E} \rightarrow \mathcal{E}'$ is a geometric morphism of ∞ -topoi, and $\eta: X' \rightarrow \pi^*X$ is a morphism of \mathcal{E}' . Now let $\mathcal{R} \subset \mathcal{Q} \times_{\mathcal{Z}_{\mathbf{Top}}} {}^L\mathbf{Top}_{\infty}^{\text{op}}$ be the full subcategory spanned by those pairs (\mathcal{E}, X) such that X is a compact as an object of \mathcal{E}_* .

For any ∞ -topos \mathcal{E} , one may identify the pullback $\mathcal{R} \times_{{}^L\mathbf{Top}_{\infty}^{\text{op}}} \{\mathcal{E}\}$ with the ∞ -category $\mathcal{E}_*^{\omega} \simeq \mathcal{R}(\mathcal{E})_{*\mathcal{E}}$. Consequently, the functor $r: \mathcal{R} \rightarrow {}^L\mathbf{Top}_{\infty}^{\text{op}}$ is an inner fibration whose fibers are, when equipped with the maximal pair structure, Waldhausen ∞ -categories.

12.17. Definition. We say that a geometric morphism $\pi^*: \mathcal{E}' \rightarrow \mathcal{E}$ is *small* if its right adjoint π_* preserves filtered colimits. Let us write ${}^L\mathbf{Top}_{\infty, \omega}^{\text{op}}$ for the subcategory of ${}^L\mathbf{Top}_{\infty}^{\text{op}}$ containing all the objects, whose morphisms are small.

12.18. Lemma. Suppose (\mathcal{E}, X) an object of \mathcal{R} . For any étale morphism of ∞ -topoi $\pi^*: \mathcal{E}'' \rightarrow \mathcal{E}$, there is an r -cocartesian edge $(\mathcal{E}, X) \rightarrow (\mathcal{E}'', X'')$ lying over π^* . Dually, for any small morphism of ∞ -topoi $\pi^*: \mathcal{E} \rightarrow \mathcal{E}'$, there is a cartesian edge $(\mathcal{E}', X') \rightarrow (\mathcal{E}, X)$ lying over (π^*, η) . In particular, the functor

$$\mathcal{R} \times_{{}^L\mathbf{Top}_{\infty}^{\text{op}}} {}^L\mathbf{Top}_{\infty, \acute{e}t}^{\text{op}} \rightarrow {}^L\mathbf{Top}_{\infty, \acute{e}t}^{\text{op}}$$

is a Waldhausen cocartesian fibration, and the functor

$$\mathcal{R} \times_{{}^L\mathbf{Top}_{\infty}^{\text{op}}} {}^L\mathbf{Top}_{\infty, \omega}^{\text{op}} \rightarrow {}^L\mathbf{Top}_{\infty, \omega}^{\text{op}}$$

is a Waldhausen cartesian fibration.

Proof. It is enough to make the following observations.

(12.18.1) If $\pi^*: \mathcal{E}'' \rightarrow \mathcal{E}$ is an étale morphism of ∞ -topoi, then composition with its left adjoint $\pi_!$ induces an exact functor

$$\mathcal{E}_*^{\omega} \simeq \mathcal{R}(\mathcal{E})_{*\mathcal{E}} \rightarrow \mathcal{R}(\mathcal{E}'')_{\pi_!(*)_{\mathcal{E}}}.$$

We may now compose this with the exact functor $\mathcal{R}(\mathcal{E}'')_{\pi_!(*)_{\mathcal{E}}} \rightarrow \mathcal{R}(\mathcal{E}'')_{*\mathcal{E}''} \simeq (\mathcal{E}'')_*^{\omega}$ guaranteed by Pr. 12.3 to construct the desired r -cocartesian edge.

(12.18.2) Dually, if $\pi^*: \mathcal{E} \rightarrow \mathcal{E}'$ is a small morphism of ∞ -topoi, then composition with π^* induces an exact functor $\mathcal{E}_*^{\omega} \simeq \mathcal{R}(\mathcal{E})_{*\mathcal{E}} \rightarrow \mathcal{R}(\mathcal{E}')_{*\mathcal{E}'} \simeq (\mathcal{E}')_*^{\omega}$. \square

12.19. Definition. The *Waldhausen A-theory of ∞ -topoi* $A: {}^L\mathbf{Top}_{\infty, \acute{e}t}^{\text{op}} \rightarrow \mathbf{Kan}_*$ is the composite $K \circ R$, where $R: {}^L\mathbf{Top}_{\infty, \acute{e}t}^{\text{op}} \rightarrow \mathbf{Wald}_{\infty}$ classifies the Waldhausen cocartesian fibration

$$\mathcal{R} \times_{{}^L\mathbf{Top}_{\infty}^{\text{op}}} {}^L\mathbf{Top}_{\infty, \acute{e}t}^{\text{op}} \rightarrow {}^L\mathbf{Top}_{\infty, \acute{e}t}^{\text{op}}.$$

The functor $A: {}^L\mathbf{Top}_{\infty, \acute{e}t}^{\text{op}} \rightarrow \mathbf{Sp}_{\geq 0}$ is the composite $\mathbf{K}^{\text{conn}} \circ R$.

Similarly, the *transfer Waldhausen A-theory of ∞ -topoi* $A^{\text{tr}}: {}^L\mathbf{Top}_{\infty, \omega}^{\text{op}} \rightarrow \mathbf{Kan}_*$ is the composite $K \circ R^{\text{tr}}$, where $R^{\text{tr}}: {}^L\mathbf{Top}_{\infty, \omega}^{\text{op}} \rightarrow \mathbf{Wald}_{\infty}$ classifies the Waldhausen cartesian fibration

$$\mathcal{R} \times_{{}^L\mathbf{Top}_{\infty}^{\text{op}}} {}^L\mathbf{Top}_{\infty, \omega}^{\text{op}} \rightarrow {}^L\mathbf{Top}_{\infty, \omega}^{\text{op}}.$$

The functor $\mathbf{A}^{\text{tr}}: {}^L\mathbf{Top}_{\infty, \omega} \rightarrow \mathbf{Sp}_{\geq 0}$ is the composite $\mathbf{K}^{\text{conn}} \circ R^{\text{tr}}$.

12.20. It is clear from the definitions that for any ∞ -topos \mathcal{E} , the functor $A_{\mathcal{E}}$ is canonically equivalent to the restriction of A to the ∞ -category $\mathcal{E} \simeq ({}^L\mathbf{Top}_{\infty, \text{ét}, \mathcal{E}})^{\text{op}}$. Similarly, $A_{\mathcal{E}}^{\text{tr}}$ is canonically equivalent to the restriction of A^{tr} to the ∞ -category $\mathcal{E}_{\omega}^{\text{op}} \simeq {}^L\mathbf{Top}_{\infty, \text{ét}, \omega, \mathcal{E}}$. Thus Waldhausen A -theory for ∞ -topoi is a genuine generalization of Waldhausen A -theory.

12.21. **Notation.** By a small abuse of notation, for any étale morphism of ∞ -topoi $\pi^*: \mathcal{E}' \rightarrow \mathcal{E}$ of ∞ -topoi, write $\pi_!: \mathbf{A}(\mathcal{E}) \rightarrow \mathbf{A}(\mathcal{E}')$ for the induced morphism, and for any small morphism $\pi^*: \mathcal{E}' \rightarrow \mathcal{E}$ of ∞ -topoi, write $\pi^*: \mathbf{A}(\mathcal{E}') \rightarrow \mathbf{A}(\mathcal{E})$ for the induced morphism.

The proof of the following analogue of the base change result of Pr. 12.12 is left to the reader.

12.22. **Proposition.** *For any pullback square*

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\psi^*} & \mathcal{F}' \\ g^* \downarrow & & \downarrow g^* \\ \mathcal{E} & \xrightarrow{\psi^*} & \mathcal{E}' \end{array}$$

of ∞ -topoi in which the geometric morphisms abusively labeled ψ^ are small and the geometric morphisms abusively labeled g^* are étale, the induced square*

$$\begin{array}{ccc} \mathbf{A}(\mathcal{E}) & \xrightarrow{\psi^*} & \mathbf{A}(\mathcal{E}') \\ g_! \downarrow & & \downarrow g_! \\ \mathbf{A}(\mathcal{F}) & \xrightarrow{\psi^*} & \mathbf{A}(\mathcal{F}') \end{array}$$

commutes.

13. EXAMPLE: CONNECTIVE ALGEBRAIC K -THEORY OF E_1 -ALGEBRAS

To any associative ring in any suitable monoidal ∞ -category we can attach its ∞ -category of modules. We may then impose suitable finiteness hypotheses on these modules and extract a K -theory spectrum. Here we identify some important examples of these K -theory spectra.

13.1. **Notation.** Suppose \mathcal{A} a presentable, monoidal ∞ -category with the property that the tensor product $\otimes: \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ preserves (small) colimits separately in each variable; assume also that \mathcal{A} is *additive* in the sense that it admits direct sums, and the resulting commutative monoids $\text{Mor}_{h\mathcal{A}}(X, Y)$ are all abelian groups. We denote by $\mathbf{Alg}(\mathcal{A})$ the ∞ -category of E_1 -algebras in \mathcal{A} , and we denote by $\mathbf{Mod}^{\ell}(\mathcal{A})$ the ∞ -category $\text{LMod}(\mathcal{A})$ defined in [41, Df. 4.2.1.13]. We have the canonical presentable fibration

$$\theta: \mathbf{Mod}^{\ell}(\mathcal{A}) \rightarrow \mathbf{Alg}(\mathcal{A})$$

[41, Cor. 4.2.3.7], whose fiber over any E_1 -algebra Λ is the stable ∞ -category $\mathbf{Mod}_{\Lambda}^{\ell}$ of left Λ -modules. Informally, we describe the objects of $\mathbf{Mod}^{\ell}(\mathcal{A})$ as pairs (Λ, E) consisting of an E_1 -algebra Λ in \mathcal{A} and a left Λ -module E .

Our aim now is to impose hypotheses on the objects of (Λ, E) and pair structures on the resulting full subcategories in order to ensure that the restriction of θ is a Waldhausen cocartesian fibration.

13.2. **Definition.** For any E_1 -algebra Λ in \mathcal{A} , a left Λ -module E will be said to be *perfect* if it satisfies the following two conditions.

(13.2.1) As an object of the ∞ -category $\mathbf{Mod}_{\Lambda}^{\ell}$ of left Λ -modules, E is compact.

(13.2.2) The functor $\mathbf{Mod}_{\Lambda}^{\ell} \rightarrow \mathcal{A}$ corepresented by E is exact.

Denote by $\mathbf{Perf}^{\ell}(\mathcal{A}) \subset \mathbf{Mod}^{\ell}(\mathcal{A})$ the full subcategory spanned by those pairs (Λ, E) in which E is perfect.

13.3. These two conditions can be more efficiently expressed by saying that E is perfect just in case the functor $\mathbf{Mod}_\Lambda^\ell \rightarrow \mathcal{A}$ corepresented by E preserves all small colimits. Note that this is *not* the same as *complete compactness*, i.e., requiring that the functor $\mathbf{Mod}_\Lambda^\ell \rightarrow \mathbf{Kan}$ corepresented by E preserves all small colimits.

13.4. **Example.** When \mathcal{A} is the nerve of the ordinary category of abelian groups, $\mathbf{Alg}(\mathcal{A})$ is the category of associative rings, and $\mathbf{Mod}^\ell(\mathcal{A})$ is the nerve of the ordinary category of pairs (Λ, E) consisting of an associative ring Λ and a left Λ -module E . An Λ -module E is perfect just in case it is (1) finitely presented and (2) projective. Thus $\mathbf{Perf}_\Lambda^\ell$ is the nerve of the ordinary category of finitely generated projective Λ -modules.

13.5. **Example.** When \mathcal{A} is the ∞ -category of connective spectra, $\mathbf{Alg}(\mathcal{A})$ can be identified with the ∞ -category of connective E_1 -rings, and $\mathbf{Mod}^\ell(\mathcal{A})$ can be identified with the ∞ -category of pairs (Λ, E) consisting of a connective E_1 -ring Λ and a connective left Λ -module E . Since the functor $\Omega^\infty: \mathbf{Sp}_{\geq 0} \rightarrow \mathbf{Kan}$ is conservative [41, Cor. 5.1.3.9] and preserves sifted colimits [41, Pr. 1.4.3.9], it follows using [41, Lm. 1.3.3.10] the second condition of Df. 13.2 amounts to the requirement that E be a projective object. Now [41, Pr. 8.2.2.6 and Cor. 8.2.2.9] guarantees that the following are equivalent for a left Λ -module E .

- (13.5.1) The left Λ -module E is perfect.
- (13.5.2) The left Λ -module E is projective, and $\pi_0 E$ is finitely generated as a $\pi_0 \Lambda$ -module.
- (13.5.3) The $\pi_0 \Lambda$ -module $\pi_0 E$ is finitely generated, and for every $\pi_0 \Lambda$ -module M and every integer $m \geq 1$, the abelian group $\mathrm{Ext}^m(E, M)$ vanishes.
- (13.5.4) There exists a finitely generated free Λ -module F such that E is a retract of F .

13.6. **Example.** The situation for modules over simplicial associative rings is nearly identical. When \mathcal{A} is the ∞ -category of simplicial abelian groups, $\mathbf{Alg}(\mathcal{A})$ can be identified with the ∞ -category of simplicial associative rings, and $\mathbf{Mod}^\ell(\mathcal{A})$ can be identified with the ∞ -category of pairs (Λ, E) consisting of a simplicial associative ring Λ and a left Λ -module E . Since the forgetful functor $\mathcal{A} \rightarrow \mathbf{Kan}$ is conservative and preserves sifted colimits, it follows that the second condition of Df. 13.2 amounts to the requirement that E be a projective object. One may show that the following are equivalent for a left Λ -module E .

- (13.6.1) The left Λ -module E is perfect.
- (13.6.2) The left Λ -module E is projective, and $\pi_0 E$ is finitely generated as a $\pi_0 \Lambda$ -module.
- (13.6.3) The $\pi_0 \Lambda$ -module $\pi_0 E$ is finitely generated, and for every $\pi_0 \Lambda$ -module M and every integer $m \geq 1$, the abelian group $\mathrm{Ext}^m(E, M)$ vanishes.
- (13.6.4) There exists a finitely generated free Λ -module F such that E is a retract of F .

13.7. **Example.** When \mathcal{A} is the ∞ -category of *all* spectra, $\mathbf{Alg}(\mathcal{A})$ is the ∞ -category of E_1 -rings, and $\mathbf{Mod}^\ell(\mathcal{A})$ is the ∞ -category of pairs (Λ, E) consisting of an E_1 -ring Λ and a left Λ -module E . Suppose Λ an E_1 -ring. The second condition of Df. 13.2 is vacuous since \mathcal{A} is stable. Hence by [41, Pr. 8.2.5.4], the following are equivalent for a left Λ -module E .

- (13.7.1) The left Λ -module E is perfect.
- (13.7.2) The left Λ -module E is contained in the smallest stable subcategory of the ∞ -category $\mathbf{Mod}_\Lambda^\ell$ of left Λ -modules that contains Λ itself and is closed under retracts.
- (13.7.3) The left Λ -module E is compact as an object of $\mathbf{Mod}_\Lambda^\ell$.
- (13.7.4) There exists a right Λ -module E^\vee such that the functor $\mathbf{Mod}_\Lambda^\ell \rightarrow \mathbf{Kan}$ informally written as $\Omega^\infty(E^\vee \otimes_\Lambda -)$ is corepresented by E .

Now we wish to endow $\mathbf{Perf}^\ell(\mathcal{A})$ with a suitable pair structure. In general, this may not be possible, but we can isolate those situations in which it is possible.

13.8. **Definition.** Denote by S the class of morphisms $(\Lambda', E') \rightarrow (\Lambda, E)$ of $\mathbf{Perf}^\ell(\mathcal{A})$ with the following two properties.

- (13.8.1) The morphism $\Lambda' \rightarrow \Lambda$ of $\mathbf{Alg}(\mathcal{A})$ is an equivalence.

(13.8.2) Any pushout diagram

$$\begin{array}{ccc} (\Lambda', E') & \longrightarrow & (\Lambda, E) \\ \downarrow & & \downarrow \\ (\Lambda', 0) & \longrightarrow & (\Lambda, E'') \end{array}$$

in $\mathbf{Mod}^\ell(\mathcal{A})$ in which $0 \in \mathbf{Mod}_{\Lambda'}^\ell$ is a zero object is also a pullback diagram, and the Λ -module E'' is perfect.

We shall say that \mathcal{A} is *admissible* if the class S is stable under pushout in $\mathbf{Perf}^\ell(\mathcal{A})$ and composition.

13.9. Example. When \mathcal{A} is the nerve of the category of abelian groups, S is the class of morphisms $(\Lambda', E') \rightarrow (\Lambda, E)$ such that $\Lambda' \rightarrow \Lambda$ is an isomorphism, and the induced map of Λ' -modules $E' \rightarrow E$ is an admissible monomorphism. It is a familiar fact that these are closed under pushout and composition, so that the nerve of the category of abelian groups is admissible.

13.10. Example. When \mathcal{A} is the ∞ -category of connective spectra or the ∞ -category of simplicial abelian groups, S is the class of morphisms $(\Lambda', E') \rightarrow (\Lambda, E)$ such that $\Lambda' \rightarrow \Lambda$ is an equivalence, and the induced homomorphism $\mathrm{Ext}^0(E, M) \rightarrow \mathrm{Ext}^0(E', M)$ is a surjection for every $\pi_0 \Lambda'$ -module M . This is visibly closed under composition. To see that these are closed under pushouts, let us proceed in two steps. First, for any morphism $\Lambda \rightarrow \Lambda'$ of $\mathbf{Alg}(\mathcal{A})$, the functor informally described as $E \mapsto E \otimes_\Lambda \Lambda'$ clearly carries morphisms of $\mathbf{Perf}_\Lambda^\ell$ that lie in S to morphisms of $\mathbf{Perf}_{\Lambda'}^\ell$ that lie in S . Now, for a fixed E_1 -algebra Λ in \mathcal{A} , suppose

$$\begin{array}{ccc} E' & \longrightarrow & E \\ \downarrow & & \downarrow \\ F' & \longrightarrow & F \end{array}$$

a pushout square in $\mathbf{Perf}_\Lambda^\ell$ in which $E' \rightarrow E$ lies in the class S , and suppose M a $\pi_0 \Lambda$ -module M . For any morphism $F' \rightarrow M$, one may precompose to obtain a morphism $E' \rightarrow M$. Our criterion on the morphism $E' \rightarrow E$ now guarantees that there is a commutative square

$$\begin{array}{ccc} E' & \longrightarrow & E \\ \downarrow & & \downarrow \\ F' & \longrightarrow & M \end{array}$$

up to homotopy. Now the universal property of the pushout yields a morphism $F \rightarrow M$ that extends the morphism $F' \rightarrow M$, up to homotopy. Thus both connective spectra and simplicial abelian groups are admissible ∞ -categories.

13.11. Example. When \mathcal{A} is the ∞ -category of all spectra, every morphism is contained in the class S . Hence the ∞ -category of all spectra is an admissible ∞ -category.

13.12. Notation. If \mathcal{A} is admissible, denote by $\mathbf{Perf}_\dagger^\ell(\mathcal{A})$ the subcategory of $\mathbf{Perf}^\ell(\mathcal{A})$ whose morphisms are those that lie in the class S . With this pair structure, the ∞ -category \mathbf{Perf}^ℓ is a Waldhausen ∞ -category.

13.13. Lemma. *If \mathcal{A} is admissible, then the functor $\mathbf{Perf}^\ell(\mathcal{A}) \rightarrow \mathbf{Alg}(\mathcal{A})$ is a Waldhausen cocartesian fibration.*

Proof. It is clear that the fibers of this cocartesian fibration are Waldhausen ∞ -categories. We claim that for any morphism $\Lambda' \rightarrow \Lambda$ of E_1 -algebras, the corresponding functor

$$\mathbf{Mod}_{\Lambda'}^\ell \rightarrow \mathbf{Mod}_\Lambda^\ell$$

given informally by the assignment $E' \mapsto \Lambda \otimes_{\Lambda'} E'$ carries perfect modules to perfect modules. Indeed, it is enough to show that the right adjoint functor $\mathbf{Mod}_\Lambda^\ell \rightarrow \mathbf{Mod}_{\Lambda'}^\ell$ preserves small colimits. This is immediate, since colimits are computed in the underlying ∞ -category \mathcal{A} [41, Pr. 3.2.3.1].

The induced functor $\mathbf{Perf}_{\Lambda'}^\ell \rightarrow \mathbf{Perf}_\Lambda^\ell$ carries a cofibration $F' \rightarrowtail E'$ to the morphism of left Λ -modules $F' \otimes_{\Lambda'} \Lambda \rightarrow E' \otimes_{\Lambda'} \Lambda$, which fits into a pushout square

$$\begin{array}{ccc} (\Lambda', F') & \longrightarrow & (\Lambda', E') \\ \downarrow & & \downarrow \\ (\Lambda, F' \otimes_{\Lambda'} \Lambda) & \longrightarrow & (\Lambda, E' \otimes_{\Lambda'} \Lambda) \end{array}$$

in $\mathbf{Perf}^\ell(\mathcal{A})$; hence $F' \otimes_{\Lambda'} \Lambda \rightarrow E' \otimes_{\Lambda'} \Lambda$ is a cofibration. \square

13.14. Definition. The *connective algebraic K-theory of E_1 -rings*, which we will abusively denote

$$\mathbf{K}^{\text{conn}}: \mathbf{Alg}(\mathcal{A}) \rightarrow \mathbf{Kan}_*,$$

is the composite functor $\mathbf{K}^{\text{conn}} \circ P$, where $P: \mathbf{Alg}(\mathcal{A}) \rightarrow \mathbf{Wald}_\infty$ is the functor classified by the Waldhausen cocartesian fibration $\mathbf{Perf}^\ell(\mathcal{A}) \rightarrow \mathbf{Alg}(\mathcal{A})$.

13.15. Construction. The preceding definition ensures that K is well-defined up to a contractible ambiguity. To obtain an explicit model of K , we proceed in the following manner. Apply \mathcal{S} to $\mathbf{Perf}^\ell(\mathcal{A}) \rightarrow \mathbf{Alg}(\mathcal{A})$ in order to obtain a Waldhausen cocartesian fibration $\mathcal{S}\mathbf{Perf}^\ell(\mathcal{A}) \rightarrow N\Delta^{\text{op}} \times \mathbf{Alg}(\mathcal{A})$. Now consider the subcategory $\iota_{N\Delta^{\text{op}} \times \mathbf{Alg}(\mathcal{A})} \mathcal{S}\mathbf{Perf}^\ell(\mathcal{A}) \subset \mathcal{S}\mathbf{Perf}^\ell(\mathcal{A})$ consisting of cocartesian edges. The composite

$$\iota_{N\Delta^{\text{op}} \times \mathbf{Alg}(\mathcal{A})} \mathcal{S}\mathbf{Perf}^\ell(\mathcal{A}) \longrightarrow N\Delta^{\text{op}} \times \mathbf{Alg}(\mathcal{A}) \longrightarrow \mathbf{Alg}(\mathcal{A})$$

is now a left fibration with a contractible space of sections given by

$$\mathbf{Alg}(\mathcal{A}) \cong \{0\} \times \mathbf{Alg}(\mathcal{A}) \xleftarrow{\sim} \iota_{\mathcal{S}_0} \mathcal{S}\mathbf{Perf}^\ell(\mathcal{A}) \hookrightarrow \iota_{N\Delta^{\text{op}} \times \mathbf{Alg}(\mathcal{A})} \mathcal{S}\mathbf{Perf}^\ell(\mathcal{A}).$$

It is clear by construction that this left fibration classifies a functor $L: \mathbf{Alg}(\mathcal{A}) \rightarrow \mathbf{Kan}_*$ such that $K \simeq \Omega \circ L$.

Let us now concentrate on the case in which \mathcal{A} is the ∞ -category of spectra.

13.16. Proposition. Suppose Λ an E_1 ring spectrum, and suppose $S \subset \pi_*\Lambda$ a collection of homogeneous elements satisfying the left Ore condition [41, Df. 8.2.4.1]. Then the morphism $\Lambda \rightarrow \Lambda[S^{-1}]$ of $\mathbf{Alg}(\mathbf{Sp})$ induces a cofiber sequence

$$\mathbf{K}^{\text{conn}}(\mathbf{Nil}_{(\Lambda, S)}^{\ell, \omega}) \rightarrow \mathbf{K}^{\text{conn}}(\Lambda) \rightarrow \mathbf{K}^{\text{conn}}(\Lambda[S^{-1}]),$$

where $\mathbf{Nil}_{(\Lambda, S)}^{\ell, \omega} \subset \mathbf{Perf}_\Lambda^\ell$ is the full subcategory spanned by those perfect left Λ -modules that are S -nilpotent.

Proof. We wish to apply Cor. 10.8.5 to the t-structure defined by the pair $(\mathbf{Nil}_{(\Lambda, S)}^\ell, \mathbf{Loc}_{(\Lambda, S)}^\ell)$, where $\mathbf{Nil}_{(\Lambda, S)}^\ell \subset \mathbf{Mod}_\Lambda^\ell$ is the full subcategory spanned by the S -nilpotent left Λ -modules, and $\mathbf{Loc}_{(\Lambda, S)}^\ell \subset \mathbf{Mod}_\Lambda^\ell$ is the full subcategory spanned by the S -local left Λ -modules. To this end, we note that $\mathbf{Mod}_\Lambda^\ell$ is compactly generated, and $\mathbf{Loc}_{(\Lambda, S)}^\ell \subset \mathbf{Mod}_\Lambda^\ell$ is in fact stable under all colimits [41, Rk. 8.2.4.16]. Now the result follows from the discussion preceding [41, Rk. 8.2.4.26]. \square

Such a result is surely well-known among experts; see for example [13, Pr. 1.4 and Pr. 1.5].

13.17. Example. For a prime p (suppressed from the notation) and an integer $n \geq 0$, the truncated Brown–Peterson spectra $\mathbf{BP}\langle n \rangle$, with coefficient ring

$$\pi_*\mathbf{BP}\langle n \rangle \cong \mathbf{Z}_{(p)}[v_1, v_2, \dots, v_n]$$

admit compatible E_1 structures [36, p. 506]. We may consider the multiplicative system $S \subset \pi_*\mathbf{BP}\langle n \rangle$ of homogeneous elements generated by v_n . Then $\mathbf{BP}\langle n \rangle[v_n^{-1}]$ is an E_1 -algebra equivalent to the Johnson–Wilson spectrum $E(n)$. The exact sequence above yields a cofiber sequence of connective spectra

$$\mathbf{K}^{\text{conn}}(\mathbf{Nil}_{(\mathbf{BP}\langle n \rangle, S)}^{\ell, \omega}) \rightarrow \mathbf{K}^{\text{conn}}(\mathbf{BP}\langle n \rangle) \rightarrow \mathbf{K}^{\text{conn}}(E(n)).$$

The content of a well-known conjecture of Ausoni–Rognes [1, (0.2)] identifies the fiber term (possibly after p -adic completion) as $\mathbf{K}^{\text{conn}}(\mathbf{BP}\langle n-1 \rangle)$. In light of results such as [41, Lm. 8.4.2.13], such a result will follow from a suitable form of a *Dévissage Theorem* [44, Th. 4]; we hope to return to such a result in later work (cf. [51, 1.11.1]).

Of course, when $n = 1$, such a Dévissage Theorem has already been provided thanks to beautiful work of Andrew Blumberg and Mike Mandell [13]. They prove that the K -theory of the ∞ -category of perfect, β -nilpotent modules over the p -local Adams summand can be identified with the K -theory of $\mathbf{Z}_{(p)}$. Consequently, they provide a cofiber sequence of connective spectra

$$\mathbf{K}^{\text{conn}}(\mathbf{Z}_{(p)}) \longrightarrow \mathbf{K}^{\text{conn}}(\ell) \longrightarrow \mathbf{K}^{\text{conn}}(L).$$

14. EXAMPLE: CONNECTIVE ALGEBRAIC K -THEORY OF DERIVED STACKS

Here we introduce the algebraic K -theory of spectral Deligne–Mumford stacks in the sense of Lurie, and we prove an easy localization theorem (analogous to what Thomason called the “Proto-localization Theorem”) in this context.

14.1. We shall appeal here to the theory of nonconnective spectral Deligne–Mumford stacks and their module theory as exposed in [39, 40]. Much of what we will say can probably be done in other contexts of derived algebraic geometry as well, such as [55, 56]; we have opted to use Lurie’s approach only because that is the one with which we are least unfamiliar. We begin by summarizing some general facts about quasicoherent modules over nonconnective spectral Deligne–Mumford stacks. Since Lurie at times concentrates on connective Deligne–Mumford stacks, we will at some points comment on how to extend the relevant definitions and results to the nonconnective case.

14.2. **Notation.** Recall from [40, §2.3, Pr. 2.5.1] that the functor $\mathbf{Sch}(\mathcal{G}_{\text{ét}}^{nM})^{\text{op}} \rightarrow \mathbf{Stk}^{\text{nc}}$ is a cocartesian fibration whose fiber over a nonconnective spectral Deligne–Mumford stack $(\mathcal{E}, \mathcal{O})$ is the stable, presentable ∞ -category $\mathbf{QCoh}(\mathcal{E}, \mathcal{O})$ of *quasicoherent* \mathcal{O} -modules.

For any nonconnective Deligne–Mumford stack $(\mathcal{E}, \mathcal{O})$, the following are equivalent for an \mathcal{O} -module \mathcal{M} .

- (14.2.1) The \mathcal{O} -module \mathcal{M} is quasicoherent.
- (14.2.2) For any morphism $U \rightarrow V$ of \mathcal{E} such that $(\mathcal{X}_{/U}, \mathcal{O}_{|U})$ and $(\mathcal{X}_{/V}, \mathcal{O}_{|V})$ are affine, the natural morphism $\mathcal{M}(V) \otimes_{\mathcal{O}(V)} \mathcal{O}(U) \rightarrow \mathcal{M}(U)$ is an equivalence.
- (14.2.3) The following conditions obtain.
 - (14.2.3.1) For every integer n , the homotopy sheaf $\pi_n \mathcal{M}$ is a quasicoherent module on the underlying ordinary Deligne–Mumford stack of $(\mathcal{E}, \mathcal{O})$
 - (14.2.3.2) The object $\Omega^\infty \mathcal{M}$ is hypercomplete in the ∞ -topos \mathcal{E} .

Using ideas from [40, §2.7], we shall now make sense of the notion of quasicoherent module over any functor $\mathbf{CAlg} \rightarrow \mathbf{Kan}(\kappa_1)$. As suggested in [40, Rk. 2.7.9], write $\mathbf{QCoh}: \text{Fun}(\mathbf{CAlg}, \mathbf{Kan}(\kappa_1))^{\text{op}} \rightarrow \mathbf{Cat}_\infty(\kappa_1)$ for the right Kan extension of the functor $\mathbf{CAlg} \rightarrow \mathbf{Cat}_\infty(\kappa_1)$ that classifies the cocartesian fibration $\mathbf{Mod} \rightarrow \mathbf{CAlg}$. Then for any functor $X: \mathbf{CAlg} \rightarrow \mathbf{Kan}(\kappa_1)$, we obtain the ∞ -category of *quasicoherent modules* $\mathbf{QCoh}(X)$ on the functor X . Many of the results of §2.7 of loc. cit. hold in this context with precisely the same proofs, including the following brace of results.

14.3. **Proposition** (cf. [40, Rk. 2.7.17]). *For any functor $X: \mathbf{CAlg} \rightarrow \mathbf{Kan}(\kappa_1)$, the ∞ -category $\mathbf{QCoh}(X)$ is stable.*

14.4. **Proposition** (cf. [40, Rk. 2.7.18]). *Suppose $(\mathcal{E}, \mathcal{O})$ a nonconnective Deligne–Mumford stack representing a functor $X: \mathbf{CAlg} \rightarrow \mathbf{Kan}(\kappa_1)$. Then there is a canonical equivalence of ∞ -categories*

$$\mathbf{QCoh}(\mathcal{E}, \mathcal{O}) \simeq \mathbf{QCoh}(X).$$

14.5. **Definition.** Suppose $X: \mathbf{CAlg} \rightarrow \mathbf{Kan}(\kappa_1)$ a functor. We say that a quasicoherent module \mathcal{M} on X is *perfect* if for any E_∞ ring A and any point $x \in X(A)$, the A -module $\mathcal{M}(x)$ is perfect. Write $\mathbf{Perf}(X) \subset \mathbf{QCoh}(X)$ for the full subcategory spanned by the perfect modules.

In particular, we can now use Pr. 14.4 to specialize the notion of perfect module to the setting of nonconnective Deligne–Mumford stacks.

14.6. **Notation.** Denote by $\mathbf{Perf} \subset \mathbf{Sch}(\mathcal{G}_{\text{ét}}^{nM})^{\text{op}}$ the full subcategory of those objects $(\mathcal{E}, \mathcal{O}, \mathcal{M})$ such that \mathcal{M} is perfect.

14.7. For any functor $X: \mathbf{CAlg} \rightarrow \mathbf{Kan}(\kappa_1)$, the ∞ -category $\mathbf{QCoh}(X)$ admits a symmetric monoidal structure [40, Nt. 2.7.27]. Moreover, this is functorial, yielding a functor

$$\mathbf{QCoh}^{\otimes}: \mathrm{Fun}(\mathbf{CAlg}, \mathbf{Kan}(\kappa_1))^{\mathrm{op}} \rightarrow \mathbf{CAlg}(\mathbf{Cat}_{\infty}(\kappa_1)).$$

14.8. **Proposition** (cf. [40, Pr. 2.7.28]). *For any functor $X: \mathbf{CAlg} \rightarrow \mathbf{Kan}(\kappa_1)$ a quasicoherent module \mathcal{M} on X is perfect if and only if it is a dualizable object of $\mathbf{QCoh}(X)$.*

Since the pullback functors are symmetric monoidal, they preserve dualizable objects. This proves the following.

14.8.1. **Corollary.** *The functor $\mathbf{Perf} \rightarrow \mathbf{Stk}^{\mathrm{nc}}$ is a cocartesian fibration.*

We endow \mathbf{Perf} with a pair structure by $\mathbf{Perf}_{\dagger} := \mathbf{Perf} \times_{\mathbf{Stk}^{\mathrm{nc}}} \iota \mathbf{Stk}^{\mathrm{nc}}$, so that the fibers are equipped with the maximal pair structure.

14.9. **Proposition.** *The functor $\mathbf{Perf} \rightarrow \mathbf{Stk}^{\mathrm{nc}}$ is a Waldhausen cocartesian fibration.*

In fact, the fiber over a nonconnective Deligne–Mumford stack $(\mathcal{E}, \mathcal{O})$ is a stable ∞ -category $\mathbf{Perf}(\mathcal{E}, \mathcal{O})$.

14.10. **Definition.** The *algebraic K-theory of nonconnective Deligne–Mumford stacks* is the functor that we abusively denote $\mathbf{K}^{\mathrm{conn}}: \mathbf{Stk}^{\mathrm{nc}} \rightarrow \mathbf{Sp}_{\geq 0}$ given by the composite $\mathbf{K}^{\mathrm{conn}} \circ P$, where P is the functor $\mathbf{Stk}^{\mathrm{nc}, \mathrm{op}} \rightarrow \mathbf{Wald}_{\infty}$ classified by the Waldhausen cocartesian fibration $\mathbf{Perf} \rightarrow \mathbf{Stk}^{\mathrm{nc}}$.

14.11. **Lemma.** *For any open immersion of quasicompact nonconnective spectral Deligne–Mumford stacks $j: \mathcal{U} \rightarrow \mathcal{X}$, the induced functor $j_{\star}: \mathbf{QCoh}(\mathcal{U}) \rightarrow \mathbf{QCoh}(\mathcal{X})$ is fully faithful.*

Proof. When \mathcal{X} is of the form $\mathrm{Spec}^{\mathrm{\acute{e}t}} A$, this is proved in [40, Cor. 2.4.6]. For any map $x: \mathrm{Spec}^{\mathrm{\acute{e}t}} A \rightarrow \mathcal{X}$, we have the open immersion $\mathcal{U} \times_{\mathcal{X}} \mathrm{Spec}^{\mathrm{\acute{e}t}} A \rightarrow \mathrm{Spec}^{\mathrm{\acute{e}t}} A$, which induces a fully faithful functor

$$\mathbf{QCoh}(\mathcal{U} \times_{\mathcal{X}} \mathrm{Spec}^{\mathrm{\acute{e}t}} A) \rightarrow \mathbf{QCoh}(\mathrm{Spec}^{\mathrm{\acute{e}t}} A).$$

Now letting A vary and applying [40, Pr. 2.4.5(3)], we obtain a functor $\mathbf{CAlg}_{\mathcal{X}/} \rightarrow \mathcal{O}(\mathbf{Cat}_{\infty}(\kappa_1))$ whose values are all fully faithful functors. Thanks to Pr. 14.4, the limit of this functor is then equivalent to a functor

$$\alpha: \lim_{A \in \mathbf{CAlg}_{\mathcal{X}/}} \mathbf{QCoh}(\mathcal{U} \times_{\mathcal{X}} \mathrm{Spec}^{\mathrm{\acute{e}t}} A) \rightarrow \mathbf{QCoh}(\mathcal{X}),$$

which is thus fully faithful. We aim to identify this functor with j_{\star} .

Since each of the ∞ -categories $\mathbf{QCoh}(\mathcal{U} \times_{\mathcal{X}} \mathrm{Spec}^{\mathrm{\acute{e}t}} A)$ can itself be described as the limit of the ∞ -categories \mathbf{Mod}_B for $B \in \mathbf{CAlg}_{\mathcal{U} \times_{\mathcal{X}} \mathrm{Spec}^{\mathrm{\acute{e}t}} A/}$, it follows that the source of α can be expressed as the limit of the ∞ -categories \mathbf{Mod}_B over the ∞ -category C of squares of nonconnective Deligne–Mumford stacks of the form

$$\begin{array}{ccc} \mathrm{Spec}^{\mathrm{\acute{e}t}} B & \longrightarrow & \mathrm{Spec}^{\mathrm{\acute{e}t}} A \\ \downarrow & & \downarrow \\ \mathcal{U} & \xrightarrow{j} & \mathcal{X}. \end{array}$$

Now there is a forgetful functor $g: C \rightarrow \mathbf{CAlg}_{\mathcal{U}/}$ that carries an object as above to the morphism $\mathrm{Spec}^{\mathrm{\acute{e}t}} B \rightarrow \mathcal{U}$. This is the functor that induces the canonical functor

$$\lim_{A \in \mathbf{CAlg}_{\mathcal{X}/}} \mathbf{QCoh}(\mathcal{U} \times_{\mathcal{X}} \mathrm{Spec}^{\mathrm{\acute{e}t}} A) \rightarrow \mathbf{QCoh}(\mathcal{U});$$

hence it suffices to show that g is right cofinal. This now follows from the fact that the functor g admits a right adjoint $\mathbf{CAlg}_{\mathcal{U}/} \rightarrow C$, which carries a morphism $x: \mathrm{Spec}^{\mathrm{\acute{e}t}} C \rightarrow \mathcal{U}$ to the object

$$\begin{array}{ccc} \mathrm{Spec}^{\mathrm{\acute{e}t}} C & = & \mathrm{Spec}^{\mathrm{\acute{e}t}} C \\ x \downarrow & & \downarrow j \circ x \\ \mathcal{U} & \xrightarrow{j} & \mathcal{X}. \end{array}$$

The proof is complete. □

14.12. Notation. For any open immersion $j: \mathcal{U} \rightarrow \mathcal{X}$ of quasicompact nonconnective spectral Deligne–Mumford stacks, let us write $\mathbf{Perf}(\mathcal{X} \setminus \mathcal{U})$ for the full subcategory of $\mathbf{Perf}(\mathcal{X})$ spanned by those perfect modules \mathcal{M} on \mathcal{X} such that $j^*\mathcal{M} \simeq 0$. Write $\mathbf{K}^{\mathrm{conn}}(\mathcal{X} \setminus \mathcal{U})$ for $\mathbf{K}^{\mathrm{conn}}(\mathbf{Perf}(\mathcal{X} \setminus \mathcal{U}))$.

Now the Special Fibration Theorem instantly yields the following.

14.13. Proposition (“Proto-localization,” cf. [51, Th. 5.1]). *For any open immersion $j: \mathcal{U} \rightarrow \mathcal{X}$ of quasicompact nonconnective spectral Deligne–Mumford stacks, the functor $j^*: \mathbf{Perf}(\mathcal{X}) \rightarrow \mathbf{Perf}(\mathcal{U})$ induces a cofiber sequence of connective spectra*

$$\mathbf{K}^{\mathrm{conn}}(\mathcal{X} \setminus \mathcal{U}) \longrightarrow \mathbf{K}^{\mathrm{conn}}(\mathcal{X}) \longrightarrow \mathbf{K}^{\mathrm{conn}}(\mathcal{U}).$$

This provides a new proof of Thomason’s theorem [51, Th. 5.1], as well as a new proof of the localization theorem for classical Deligne–Mumford stacks offered by Krishna and Østvær [35, Th. 3.7].

When j is the open complement of a closed immersion $i: \mathcal{Z} \rightarrow \mathcal{X}$, one may ask whether $\mathbf{K}^{\mathrm{conn}}(\mathcal{X} \setminus \mathcal{U})$ can be identified with $\mathbf{K}^{\mathrm{conn}}(\mathcal{Z})$. In general, the answer is no, but in special situations, such an identification is possible. Classically, this is the result of a *Dévissage Theorem* [44, Th. 4]; we hope to return to a higher categorical analogue of such a result in later work (cf. [51, 1.11.1])

REFERENCES

1. C. Ausoni and J. Rognes, *Algebraic K-theory of topological K-theory*, Acta Math. **188** (2002), no. 1, 1–39. MR MR1947457 (2004f:19007)
2. C. Barwick, *Homotopy coherent algebra I. Operator categories*, Available electronically from <http://www.math.mit.edu/~clarkbar/>.
3. C. Barwick and D. M. Kan, *Partial model categories and their simplicial nerves*, arXiv:1102.2512v1, 2011.
4. ———, *Relative categories: Another model for the homotopy theory of homotopy theories*, Indag. Math. **23** (2012), no. 1–2, 42–68.
5. C. Barwick and C. Schommer-Pries, *On the unicity of the homotopy theory of higher categories*, Preprint arXiv:1112.0040v1, December 2011.
6. J. E. Bergner, *A characterization of fibrant Segal categories*, Proc. Amer. Math. Soc. **135** (2007), no. 12, 4031–4037 (electronic). MR 2341955 (2008h:55031)
7. ———, *A model category structure on the category of simplicial categories*, Trans. Amer. Math. Soc. **359** (2007), no. 5, 2043–2058. MR 2276611 (2007i:18014)
8. ———, *Three models for the homotopy theory of homotopy theories*, Topology **46** (2007), no. 4, 397–436. MR 2321038 (2008e:55024)
9. ———, *Complete Segal spaces arising from simplicial categories*, Trans. Amer. Math. Soc. **361** (2009), no. 1, 525–546. MR 2439415 (2009g:55033)
10. ———, *A survey of $(\infty, 1)$ -categories*, Towards higher categories, IMA Vol. Math. Appl., vol. 152, Springer, New York, 2010, pp. 69–83. MR 2664620 (2011e:18001)
11. P. Berthelot, A. Grothendieck, and L. Illusie, *Théorie des intersections et théorème de Riemann-Roch*, Séminaire de Géométrie Algébrique du Bois Marie 1966–67 (SGA 6). Avec la collaboration de D. Ferrand, J. P. Jouanolou, O. Jussila, S. Kleiman, M. Raynaud, J. P. Serre. Lecture Notes in Mathematics, Vol. 225, Springer-Verlag, Berlin, 1966–67. MR 50 #7133
12. A. J. Blumberg, D. Gepner, and G. Tabuada, *A universal characterization of higher algebraic K-theory*, arXiv:1001.2282v3.
13. A. J. Blumberg and M. A. Mandell, *The localization sequence for the algebraic K-theory of topological K-theory*, Acta Math. **200** (2008), no. 2, 155–179. MR 2413133 (2009f:19003)
14. ———, *Algebraic K-theory and abstract homotopy theory*, Adv. Math. **226** (2011), no. 4, 3760–3812. MR 2764905 (2012a:19007)
15. J. M. Boardman and R. M. Vogt, *Homotopy invariant algebraic structures on topological spaces*, Springer-Verlag, Berlin, 1973, Lecture Notes in Mathematics, Vol. 347. MR MR0420609 (54 #8623a)
16. D.-C. Cisinski, *Invariance de la K-théorie par équivalences dérivées*, J. K-Theory **6** (2010), no. 3, 505–546. MR 2746284
17. W. G. Dwyer, P. S. Hirschhorn, D. M. Kan, and J. H. Smith, *Homotopy limit functors on model categories and homotopical categories*, Mathematical Surveys and Monographs, vol. 113, American Mathematical Society, Providence, RI, 2004. MR MR2102294
18. W. G. Dwyer and D. M. Kan, *Calculating simplicial localizations*, J. Pure Appl. Algebra **18** (1980), no. 1, 17–35. MR 81h:55019
19. ———, *Function complexes in homotopical algebra*, Topology **19** (1980), no. 4, 427–440. MR 81m:55018
20. ———, *Simplicial localizations of categories*, J. Pure Appl. Algebra **17** (1980), no. 3, 267–284. MR 81h:55018
21. W. G. Dwyer, M. Weiss, and B. Williams, *A parametrized index theorem for the algebraic K-theory Euler class*, Acta Math. **190** (2003), no. 1, 1–104. MR 1982793 (2004d:19004)
22. A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May, *Rings, modules, and algebras in stable homotopy theory*, Mathematical Surveys and Monographs, vol. 47, American Mathematical Society, Providence, RI, 1997, With an appendix by M. Cole. MR 97h:55006

23. T. G. Goodwillie, *Calculus. I. The first derivative of pseudoisotopy theory*, *K-Theory* **4** (1990), no. 1, 1–27. MR 1076523 (92m:57027)
24. ———, *The differential calculus of homotopy functors*, Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990) (Tokyo), Math. Soc. Japan, 1991, pp. 621–630. MR 1159249 (93g:55015)
25. ———, *Calculus. II. Analytic functors*, *K-Theory* **5** (1991/92), no. 4, 295–332. MR 1162445 (93i:55015)
26. ———, *Calculus. III. Taylor series*, *Geom. Topol.* **7** (2003), 645–711 (electronic). MR 2026544 (2005e:55015)
27. A. Grothendieck, *La théorie des classes de Chern*, *Bull. Soc. Math. France* **86** (1958), 137–154. MR 0116023 (22 #6818)
28. A. Hirschowitz and C. Simpson, *Descente pour les n -champs (Descent for n -stacks)*, Preprint available from arXiv:math/9807049v3.
29. R. Joshua, *Higher intersection theory on algebraic stacks. I*, *K-Theory* **27** (2002), no. 2, 133–195. MR MR1942183 (2003k:14019a)
30. ———, *Higher intersection theory on algebraic stacks. II*, *K-Theory* **27** (2002), no. 3, 197–244. MR MR1949305 (2003k:14019b)
31. ———, *Riemann-Roch for algebraic stacks. I*, *Compositio Math.* **136** (2003), no. 2, 117–169. MR MR1967388 (2004e:14008)
32. A. Joyal, *Notes on quasi-categories*, Preprint. www.math.uchicago.edu/~may/IMA/JOYAL/, December 2008.
33. ———, *The theory of quasi-categories and its applications*, Advanced Course on Simplicial Methods in Higher Categories, vol. 2, Centre de Recerca Matemàtica, 2008.
34. A. Joyal and M. Tierney, *Quasi-categories vs Segal spaces*, Categories in algebra, geometry and mathematical physics, *Contemp. Math.*, vol. 431, Amer. Math. Soc., Providence, RI, 2007, pp. 277–326. MR 2342834 (2008k:55037)
35. A. Krishna and P. A. Østvær, *Nisnevich descent for K -theory of Deligne-Mumford stacks*, *Journal of K-Theory* **FirstView** (2011), 1–41.
36. A. Lazarev, *Towers of MU -algebras and the generalized Hopkins-Miller theorem*, *Proc. London Math. Soc.* (3) **87** (2003), no. 2, 498–522. MR 1990937 (2004c:55006)
37. J. Lurie, *Higher topos theory*, *Annals of Mathematics Studies*, no. 170, Princeton University Press, 2009, Also available from <http://www.math.harvard.edu/~lurie>.
38. ———, *$(\infty, 2)$ -categories and the Goodwillie calculus I*, Available from <http://www.math.harvard.edu/~lurie>, October 2009.
39. ———, *Derived algebraic geometry V. Structured spaces*, Available from <http://www.math.harvard.edu/~lurie>, February 2011.
40. ———, *Derived algebraic geometry VIII. Quasi-coherent sheaves and Tannaka duality theorems*, Available from <http://www.math.harvard.edu/~lurie>, May 2011.
41. ———, *Higher algebra*, Available from <http://www.math.harvard.edu/~lurie>, February 2012.
42. M. A. Mandell, *An inverse K -theory functor*, *Doc. Math.* **15** (2010), 765–791. MR 2735988 (2012a:19005)
43. R. McCarthy, *On fundamental theorems of algebraic K -theory*, *Topology* **32** (1993), no. 2, 325–328. MR 1217072 (94e:19002)
44. D. Quillen, *Higher algebraic K -theory. I*, Algebraic K -theory, I: Higher K -theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), Springer, Berlin, 1973, pp. 85–147. Lecture Notes in Math., Vol. 341. MR MR0338129 (49 #2895)
45. C. Rezk, *A model for the homotopy theory of homotopy theory*, *Trans. Amer. Math. Soc.* **353** (2001), no. 3, 973–1007 (electronic). MR MR1804411 (2002a:55020)
46. S. Schwede, *Stable homotopical algebra and Γ -spaces*, *Math. Proc. Cambridge Philos. Soc.* **126** (1999), no. 2, 329–356. MR MR1670249 (2000b:55005)
47. G. Segal, *Categories and cohomology theories*, *Topology* **13** (1974), 293–312. MR 50 #5782
48. C. Simpson, *Homotopy theory of higher categories*, *New Mathematical Monographs*, no. 19, Cambridge University Press, 2011.
49. R. E. Staffeldt, *On fundamental theorems of algebraic K -theory*, *K-Theory* **2** (1989), no. 4, 511–532. MR 990574 (90g:18009)
50. G. Tabuada, *Higher K -theory via universal invariants*, *Duke Math. J.* **145** (2008), no. 1, 121–206. MR 2451292 (2009j:18014)
51. R. W. Thomason and T. Trobaugh, *Higher algebraic K -theory of schemes and of derived categories*, The Grothendieck Festschrift, Vol. III, *Progr. Math.*, vol. 88, Birkhäuser Boston, Boston, MA, 1990, pp. 247–435. MR 92f:19001
52. B. Toën, *Théorèmes de Riemann-Roch pour les champs de Deligne-Mumford*, *K-Theory* **18** (1999), no. 1, 33–76. MR 2000h:14010
53. ———, *Vers une axiomatisation de la théorie des catégories supérieures*, *K-Theory* **34** (2005), no. 3, 233–263. MR 2182378 (2006m:55041)
54. B. Toën and G. Vezzosi, *A remark on K -theory and S -categories*, *Topology* **43** (2004), no. 4, 765–791. MR 2 061 207
55. ———, *Homotopical algebraic geometry. I. Topos theory*, *Adv. Math.* **193** (2005), no. 2, 257–372. MR MR2137288
56. ———, *Homotopical algebraic geometry. II. Geometric stacks and applications*, *Mem. Amer. Math. Soc.* **193** (2008), no. 902, x+224. MR MR2394633
57. F. Waldhausen, *Algebraic K -theory of spaces*, Algebraic and geometric topology (New Brunswick, N.J., 1983), *Lecture Notes in Math.*, vol. 1126, Springer, Berlin, 1985, pp. 318–419. MR 86m:18011
58. F. Waldhausen, B. Jähren, and J. Rognes, *The stable parametrized h -cobordism theorem*, Unpublished manuscript, 2006.
59. I. Zakharevich, *Scissors congruence as K -theory*, arXiv:1101.3833v1.

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